$\mathbb{R}, \leq \;,\; \emptyset \neq A \subseteq \mathbb{R}$

$\sup A = x \in \mathbb{R}$ 1) $\forall a \in A \quad a \leq x$

2) $\forall \varepsilon > 0 \; \exists a_\varepsilon \in A \quad a_\varepsilon > x - \varepsilon$

$\inf A = x \in \mathbb{R}$ 1) $\forall a \in A \quad x \leq a$

2) $\forall \varepsilon > 0 \; \exists a_\varepsilon \in A \quad a_\varepsilon < x + \varepsilon$

A ограничен одозго $\Rightarrow \exists \sup A \in \mathbb{R}$

A није огран одозго $\Rightarrow \nexists \sup A$

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$A = \{ q \in \mathbb{Q} \mid q^2 \leq 2 \} \longrightarrow \sup_{\mathbb{R}} A = \sqrt{2} \in \mathbb{R}$ када посматрамо у $\mathbb{R}$

$\nexists \sup_{\mathbb{Q}} A$ јер $\sqrt{2} \notin \mathbb{Q}$

* Одредити sup, inf, min, max ако постоје.

1) $A_1 = \{ \underbrace{\sqrt{n+1} - \sqrt{n}}_{a_n} \mid n \in \mathbb{N} \}$

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$a_n = (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$\inf A = 0$
$\sup A = a_1 = \sqrt{2} - 1$?

$\inf A_1 = 0$: , $0 \notin A_1 \Rightarrow \nexists \min A_1$

1) $\forall n \in \mathbb{N} \quad a_n \geq 0$ ✓ $\quad \sqrt{n+1} \geq \sqrt{n}$ ✓

2) $\varepsilon > 0$ произвољно , $\varepsilon < 1$

$n_0 = ? \qquad a_{n_0} < 0 + \varepsilon = \varepsilon$

$\sqrt{n_0+1} - \sqrt{n_0} < \varepsilon$

$\sqrt{n_0 + 1} < \varepsilon + \sqrt{n_0} \quad \uparrow^2$

$\cancel{n_0} + 1 < \varepsilon^2 + \cancel{n_0} + 2\varepsilon\sqrt{n_0}$

$2\varepsilon\sqrt{n_0} > 1 - \varepsilon^2$

$n_0 > \left(\frac{1-\varepsilon^2}{2\varepsilon}\right)^2 \Rightarrow n_0 = \left[\left(\frac{1-\varepsilon^2}{2\varepsilon}\right)^2\right] + 1 \in \mathbb{N}.$

$\sup A_1 = a_1 = \sqrt{2} - 1$:

1) $\forall n \in \mathbb{N} \quad a_1 \geq a_n$

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$a_n < a_{n-1}$

? $\sqrt{n+1} - \sqrt{n} < \sqrt{n} - \sqrt{n-1}$

$a_n \downarrow$

? $\sqrt{n+1} + \sqrt{n-1} < 2\sqrt{n} \quad \uparrow^2$

$a_1 = \max A_1$

$\cancel{n} + \cancel{1} + \cancel{n} - \cancel{1} + 2\sqrt{n^2-1} < \cancel{4}n^2$

$\Downarrow$

? $2\sqrt{n^2-1} < 2n = \sqrt{n^2}$ ✓

$\sup A_1 = \max A_1 = \sqrt{2} - 1$

② $A_2 = \left\{ \frac{m}{n} + \frac{n}{k} + \frac{k}{m} \;\middle|\; m, n, k \in \mathbb{N} \right\}$

$\sup A_2$ не постоји .... ?

$\inf A_2 = 3$

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$\inf A_2 = 3$

1) $\forall m, n, k \in \mathbb{N} \quad \frac{m}{n} + \frac{n}{k} + \frac{k}{m} \geq 3$? ✓

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$$\frac{\frac{m}{n} + \frac{n}{k} + \frac{k}{m}}{3} \geq \sqrt[3]{\frac{\cancel{m}}{\cancel{n}} \cdot \frac{\cancel{n}}{\cancel{k}} \cdot \frac{\cancel{k}}{\cancel{m}}} = \sqrt[3]{1} = 1 \quad ✓$$

$m = n = k \qquad \frac{m}{n} + \frac{n}{k} + \frac{k}{m} = 3 \in A_2$

$\min A_2 = 3$.

$\nexists \sup A_2$:

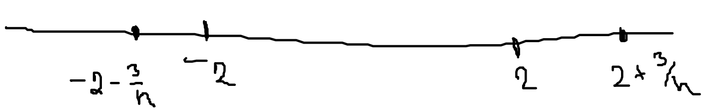
A_2 није ограничен одозго ? $\qquad \forall M \in \mathbb{R} \;\; \exists a \in A_2 \quad a > M$

$M > 0$ произвољно $\qquad a = \frac{m}{n} + \frac{n}{k} + \frac{k}{m} > M$: $\quad m = [M] + 1 \in \mathbb{N}$, $n = 1$, $k = 1$

$$a = [M] + 1 + 1 + \frac{1}{[M]+1} > M.$$

$\Rightarrow A_2$ није огран одозго $\Rightarrow \nexists \sup A_2$

③ $A_3 = \left\{ \underbrace{(-1)^{n-1}\left(2 + \frac{3}{n}\right)}_{a_n} \;\middle|\; n \in \mathbb{N} \right\}$



$\inf A_3 = -2 - \frac{3}{2}$

$\sup A_3 = 5$

n - парно $\qquad a_n = -2 - \frac{3}{n}$

n - непарно $\qquad a_n = 2 + \frac{3}{n}$

$\inf A_3 = -2 - 3/2$: $\qquad n$ - непар. $a_n > 0 > -2 - 3/2$

n - парно $\qquad a_n = -\cancel{2} - \frac{3}{n} \geq -\cancel{2} - \frac{3}{2} \Rightarrow n \geq 2$

$\Rightarrow \inf A_3 = \min A_3$

$\sup A_3 = 5$: n-чётно $a_n < 0 < 5$

n-нечётно $a_n \leq 5 \Rightarrow 2 + \frac{3}{n} \leq 5$

$\frac{3}{n} \leq 3$, $n \geq 1$ ✓

$\sup A_3 = \max A_3 = 5$

④ $A_4 = \{ \underbrace{\frac{[nx]}{n}}_{a_n} \mid n \in \mathbb{N} \}$ $x > 0$ фиксиран

$\sup A_4 = x$ $a_n = \frac{[nx]}{n} = \frac{nx - \{nx\}}{n} = x - \frac{\{nx\}}{n}$

$\inf A_4 = [x] = \min A_4$

$\underline{\underline{x-1}} \leq x - \frac{1}{n} = \frac{nx-1}{n} \leq a_n$

$\min A_4 = [x]$:

$x = [x] + \{x\} \dashrightarrow [nx] = [\underbrace{n \cdot [x]}_{\in \mathbb{Z}} + n \cdot \{x\}] = n \cdot [x] + [n\{x\}]$

$a_n = \frac{[nx]}{n} = \frac{n \cdot [x] + [n\{x\}]}{n} = [x] + \frac{\overbrace{[n\{x\}]}^{\geq 0}}{n} \geq [x] > x - 1$

$a_1 = [x]$

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$\sup A_4 = x$:

1) $\forall n \in \mathbb{N}$ $a_n = x - \frac{\overbrace{\{nx\}}^{1,0}}{n} \leq x$

2) $\varepsilon > 0$ произволно

? $\exists n_0 \in \mathbb{N}$ $a_{n_0} > \underline{\underline{x - \varepsilon}}$?

$\frac{[n_0 x]}{n_0} > x - \varepsilon$ $/ \cdot n_0$

$[n_0 x] > n_0 x - n_0 \varepsilon$

$n_0 \varepsilon > \underbrace{n_0 x - [n_0 x]}_{< 1}$?

Доволно да $n_0 \varepsilon > 1 > n_0 x - [n_0 x]$.

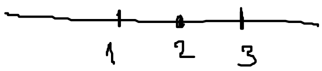
$n_0 = \left[\frac{1}{\varepsilon}\right] + 1$.

$\Rightarrow$ $x = \sup A_4$

(5) $A_5 = \{ x \in \mathbb{R} \mid x^2 \log_2|x-2| \leq 0 \}$

$x^2 \geq 0 \quad , \quad x \neq 0 \quad \Rightarrow \quad \log_2|x-2| \leq 0 = \log_2 1$

$|x-2| \leq 1 \quad \Rightarrow \quad x \in [1,3]$



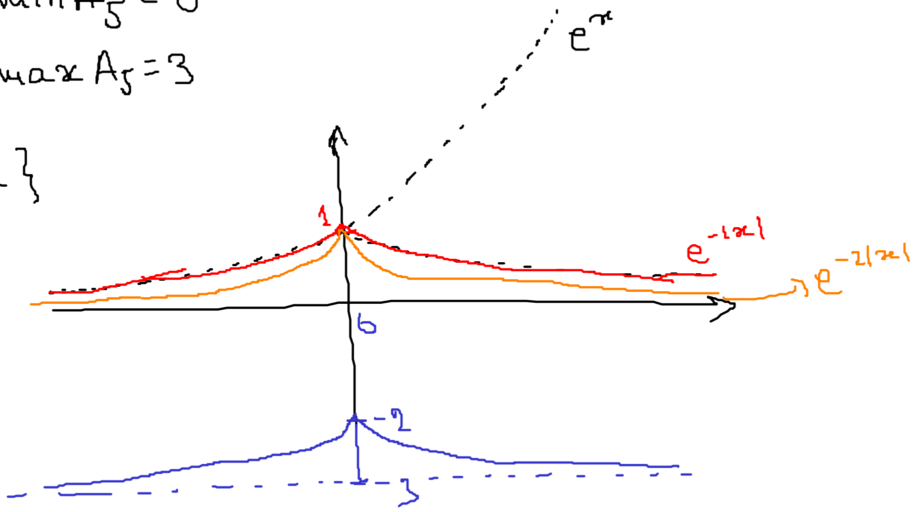
$\Rightarrow A_5 = [1,3] \cup \{0\} \quad \Rightarrow \quad \min A_5 = 0$

$\max A_5 = 3$

(6) $A_6 = \{ e^{-2|x|} - 3 \mid x \in \mathbb{R} \}$

$\inf A_6 = -3$

$\sup A_6 = -2 \quad = \max A_6$



$\sup A_6 = -2$:

1) $\forall x \in \mathbb{R} \quad e^{-2|x|} - 3 \leq -2 \quad / +3$

$e^{-2|x|} \leq 1 \quad / \ \uparrow^{-1}$

$e^{2|x|} \geq 1 \quad \checkmark$ јер $|x| \geq 0$

$\inf A_6 = -3$ 1) $\forall x \in \mathbb{R} \quad e^{-2|x|} - 3 \geq -3$

$e^{-2|x|} \geq 0 \quad \checkmark$

2) $\varepsilon > 0$ произвољно $\ ? \ \exists x_\varepsilon \in \mathbb{R} \ : \ e^{-2|x_\varepsilon|} - 3 < -3 + \varepsilon$

$\varepsilon < 1$

$e^{-2|x_\varepsilon|} < \varepsilon \quad / \ln$

$-2|x_\varepsilon| < \ln \varepsilon$

$|x_\varepsilon| > -\frac{1}{2}\ln\varepsilon > 0$

$x_\varepsilon = -\frac{1}{2}\ln\varepsilon + 1 > 0$

$$⑦ \quad A_7 = \left\{ \frac{m^2+m}{m^2-n^2} - \frac{1}{n} \;\middle|\; m>n;\ n,m\in\mathbb{N} \right\}$$

фиксирано n: $\quad b_m(n) = \dfrac{m^2+m}{m^2-n^2} - \dfrac{1}{n}$

$$\rightarrow A_n = \{ b_m^n \mid m>n \} \longrightarrow \begin{matrix}\inf A_n \\ \sup A_n\end{matrix}$$

$\sup A_7 = ?$

$$A_7 = \bigcup_{n=1}^{\infty} A_n \longrightarrow \begin{matrix}\sup A_7 = \sup\{\sup A_n \mid n\in\mathbb{N}\} = S \\ \inf A_7 = \inf\{\inf A_n \mid n\in\mathbb{N}\} = i\end{matrix}$$

$$b_m^n = \frac{m^2+m}{m^2-n^2} - \cancel{\frac{1}{n}} \overset{?}{\leq} b_{m+1}^n = \frac{(m+1)^2+(m+1)}{(m+1)^2-n^2} - \cancel{\frac{1}{n}}$$

$$\Leftrightarrow \frac{m\cdot\cancel{(m+1)}}{m^2-n^2} \overset{?}{\leq} \frac{\cancel{(m+1)}(m+2)}{(m+1)^2-n^2} \quad /: m+1>0$$

$$m\cdot(\underbrace{(m+1)^2}_{m^2+2m+1}-n^2) \leq (m^2-n^2)\cdot(m+2)$$

$$m(\cancel{m^2}+\cancel{2m}+1-\cancel{n^2}) \leq m\cdot(\cancel{m^2}-\cancel{n^2}) + 2(\cancel{m^2}-n^2)$$

$m+2n^2 \leq 0$ — ово не важи никада

$\Rightarrow b_m^n > b_{m+1}^n \longrightarrow b_m^n \downarrow$ по m

$\inf A_n = 1-\frac{1}{n}$

$\sup A_n = b_{n+1}^n = \max A_n$ јер $b_{m+1}^n < b_m^n < b_{n+1}^n$ јер је $m>n$

1) $1-\frac{1}{n} \leq b_m^n \quad \forall m>n$

$$1-\cancel{\frac{1}{n}} \leq \frac{m^2+m}{m^2-n^2} - \cancel{\frac{1}{n}} \ ? \quad \text{за } m>n$$

$$1 \leq \frac{m^2+m}{m^2-n^2} = 1 + \underbrace{\frac{m+n^2}{m^2-n^2}}_{>0} \quad \checkmark$$

2) $\forall \varepsilon>0 \ \exists m_0\in\mathbb{N} \quad b_{m_0}^n < 1-\frac{1}{n}+\varepsilon \ ?$

$$\frac{m_0^2+m_0}{m_0^2-n^2} - \cancel{\frac{1}{n}} < 1 - \cancel{\frac{1}{n}} + \varepsilon$$

$$\cancel{1} + \frac{m_0+n^2}{m_0^2-n^2} < \cancel{1} + \varepsilon \longrightarrow m_0+n^2 < \varepsilon\cdot(m_0^2-n^2)$$

$$\varepsilon\cdot m_0^2 - m_0 - \varepsilon\cdot n^2 - n^2 > 0$$

$$\varepsilon x^2 - x - (\varepsilon n^2 + n^2) > 0$$

$$x_{1/2} = \frac{1 \pm \sqrt{1 + 4\varepsilon \cdot (\varepsilon n^2 + n^2)}}{2\varepsilon}$$

$$n_0 > \frac{1 + \sqrt{1 + 4\varepsilon(\varepsilon n^2 + n^2)}}{2\varepsilon} \to n_0 = \left[\frac{1 + \sqrt{\cdots}}{2\varepsilon} \right] + 1$$

... сл. час наставак ...