① $\underbrace{\sum_{n=1}^{\infty} (-1)^n \left(\frac{4^n \cdot (n!)^2}{(2n+1)!} \right)^p}_{a_n}$

$$\frac{4^n (n!)^2}{(2n+1)!}$$

- Стирлингова ф-ла : $n! \sim \frac{n^n}{e^n}\sqrt{2n\pi}$, $n \to +\infty$

$$\frac{4^n (n!)^2}{(2n+1)!} \sim \frac{\overset{2^{2n}}{4^n} \frac{n^{2n}}{e^{2n}} \cdot 2n\pi}{\frac{(2n+1)^{2n+1}}{e^{2n+1}} \cdot \sqrt{(2n+1)\cdot 2\pi}} = \underbrace{\frac{(2n)^{2n+1}}{(2n+1)^{2n+1}}}_{<1} \cdot e \cdot \frac{\pi}{\sqrt{(2n+1)\cdot 2\pi}}$$

$$\circledast \left(\frac{4^n (n!)^2}{(2n+1)!} \right)^p \sim \underbrace{\left(1 - \frac{1}{2n+1} \right)^{(2n+1)\cdot p}}_{\substack{\downarrow \\ \frac{1}{e^p}}} \cdot \frac{e^p \cdot \pi^p}{(2n+1)^{p/2} \cdot (2\pi)^{p/2}} \sim \frac{c}{n^{p/2}}$$

$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^{p/2}}$ конв акко $p/2 > 1$

$\sum a_n$ aпс. конв. акко $p/2 > 1$, тј. $p > 2$.

$p \leq 0$ $\circledast$ $|a_n| \not\to 0$ јер $\frac{c}{(2n+1)^{p/2}} \not\to 0$ $\Rightarrow$ $\lim_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum a_n$ дивергира

- $0 < p < 2$:

$\lim_{n\to\infty} |a_n| = 0$ (на основу $\circledast$)

$|a_n| \downarrow$?

$$\frac{|a_n|}{|a_{n+1}|} = \left(\frac{\frac{4^n \, (n!)^2}{(2n+1)!}}{\frac{4^{n+1} \, ((n+1)!)^2}{(2n+3)!}} \right)^p = \left(\frac{(2n+2)\cdot(2n+3)}{4 (n+1)^2} \right)^p = \left(\frac{2n+3}{2n+2} \right)^p > 1$$

$p > 0$

$\Rightarrow |a_n| \downarrow$

Лајбниц $\Rightarrow$ $\sum_{n=1}^{\infty} (-1)^n |a_n|$ конвергира.

(2) $a_n > 0$, $\uparrow$, $\lim\limits_{n\to\infty} \dfrac{a_n}{a_1 \cdot a_2 \cdots a_{n-1}} = C > 0$, $\lim\limits_{n\to\infty} a_n = +\infty$

Да ли конв. $\sum\limits_{n=1}^{\infty} \dfrac{1}{a_n}$?

$\lim\limits_{n\to+\infty} a_n = +\infty \Rightarrow \exists n_0 : \underline{\underline{n \geq n_0}} \quad a_n > 1$

$\dfrac{a_n}{a_1 \cdots a_{n-1}} = C \Rightarrow \exists n_1 > n_0 : \underset{n \geq n_1}{} \quad \dfrac{a_n}{a_1 \cdots a_{n-1}} > \dfrac{c}{2} > 0 \quad \left(\varepsilon = \dfrac{c}{2}\right)$

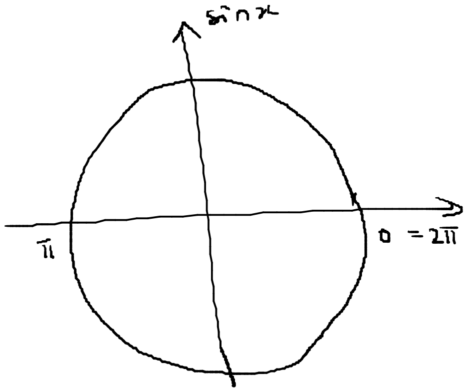
$$a_n > \frac{c}{2} \cdot \underbrace{a_1 \cdot a_2 \cdots a_{n_0-1}}_{A>0} \, \underbrace{a_{n_0} \cdots a_{n-1}}_{\geq a_{n_0}^{n-\cancel{1}-n_0+1}}$$

$$a_n \geq \frac{C \cdot A}{2} \frac{1}{a_{n_0}^{n_0}} a_{n_0}^n = C_2 \cdot a_{n_0}^n$$

$$\frac{1}{a_n} \leq \frac{1}{C_2} \cdot \left(\frac{1}{a_{n_0}}\right)^n, \quad n \geq n_1 > n_0$$

$$\sum_{n=1}^{\infty} \frac{1}{c_2} \left(\frac{1}{a_{n_0}}\right)^n = \frac{1}{c_2} \sum_{n=1}^{\infty} \left(\frac{1}{a_{n_0}}\right)^n, \quad q = \frac{1}{a_{n_0}} < 1 \Rightarrow \sum q^n \text{ конв.}$$

$\underline{\text{I}}$ поредб. крит. $\Rightarrow$ $\sum\limits_{n=1}^{\infty} \dfrac{1}{a_n}$ конв.



(3) $$\sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt{n}} \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{\sqrt{x}}\, dx}_{a_n}$$

$\sin x > 0$ на $(2k\pi, (2k+1)\pi)$

$\sin x < 0$ $((2k+1)\pi, (2k+2)\pi)$

$$a_n = (-1)^n \cdot \underbrace{\frac{1}{\sqrt{n}} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{\sqrt{x}}\, dx}_{>0}$$

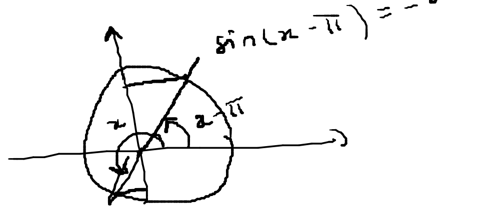
• $|a_n| \downarrow$

$$\frac{1}{\sqrt{n+1}} \underbrace{\int_{(n+1)\pi}^{(n+2)\pi} \frac{|\sin x|}{\sqrt{x}}\, dx}_{} \underset{?}{\leq} \frac{1}{\sqrt{n}} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{\sqrt{x}}\, dx$$

$t = x - \pi$

$dx = dt$

$|\sin x| = |\sin(x-\pi)| = |\sin t|$



$$\frac{1}{\sqrt{n+1}} \underbrace{\int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{\sqrt{t+\pi}}\,dt} \le \frac{1}{\sqrt{n}} \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{\sqrt{t}}\,dt \quad \checkmark$$

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

$$\frac{1}{\sqrt{t+\pi}} < \frac{1}{\sqrt{t}} \Rightarrow 0 < \frac{|\sin t|}{\sqrt{t+\pi}} < \frac{|\sin t|}{\sqrt{t}} \quad \Big/ \int_{n\pi}$$

$|a_n| \downarrow$

$|a_n| \to 0$?

$$0 \le \frac{1}{\sqrt{n}} \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{\sqrt{t}}\,dt \le \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n\pi}} \underbrace{\int_{n\pi}^{(n+1)\pi} |\sin t|\,dt} = \frac{1}{n\sqrt{\pi}} \cdot 2$$

$$\frac{1}{\sqrt{t}} \le \frac{1}{\sqrt{n\pi}}$$

$$\frac{1}{n\sqrt{\pi}} \cdot 2 \to 0$$

$$\int_0^{\pi} |\sin t|\,dt = 2$$

$$|\sin(t-\pi)| = |\sin t|$$

$T = 2\pi$

$\to 0$

Лајбниц $\Rightarrow \sum a_n$ конвергира

$$|a_n| = \frac{1}{\sqrt{n}} \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{\sqrt{t}}\,dt \ge \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{(n+1)\pi}} \underbrace{\int_{n\pi}^{(n+1)\pi} |\sin t|\,dt}_{=2}$$

$$\frac{1}{\sqrt{t}} \ge \frac{1}{\sqrt{(n+1)\pi}}$$

$$\frac{2}{\sqrt{n} \cdot \sqrt{(n+1)\cdot\pi}} \sim \frac{2}{\pi} \cdot \frac{1}{n}, \quad n \to +\infty$$

$$\sum \frac{1}{n} \text{ див.} \xrightarrow[\text{крит.}]{\text{I пореди.}} \sum |a_n| \text{ див.}$$

④ $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^2}$, $f(n)$ број цифара броја n

$f(n) = k \iff 10^{k-1} \le n \le 10^k - 1 \to n+1 \le 10^k$

10^{k-1} – најмањи природан бр. који има k-цифара

$10^k - 1$ – највећи са k циф.

$\Rightarrow k-1 \le \log_{10} n$, $k \ge \log_{10}(n+1)$

$\log_{10}(n+1) \le k \le \log_{10} n + 1$

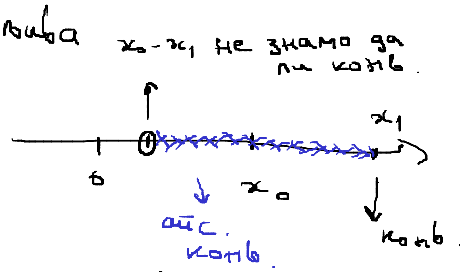
$$\rightsquigarrow \sum \frac{f(n)}{n^2} \le \sum \frac{\log_{10} n + 1}{n^2} \overset{(*)}{=} \underbrace{\sum \frac{\log_{10} n}{n^2}}_{\text{конв.}} + \underbrace{\sum \frac{1}{n^2}}_{\text{конв.}}$$

I поредб. крит. $\Rightarrow$ $\sum \frac{f(n)}{n^2}$ конв.

Степени редови

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = F(x) \quad , \quad a_n, x_0 \in \mathbb{R} \quad , \quad x \text{ - променљива}$$

$x = ?$
$F(x) \in \mathbb{R}$?



Лема: Ако $F(x)$ конв. за $x_1 \neq x_0$, онда он конв. апсолутно за x тд $|x-x_0| < |x_1 - x_0|$.

Радијус конв. $R = \sup\{ |x-x_0| : F(x) \text{ конв.} \} \geq 0$

$|x-x_0| > R \Rightarrow F(x)$ дивергира

$|x-x_0| < R \Rightarrow F(x)$ апс. конв.

$|x-x_0| = R \rightsquigarrow$ проверавамо конв.

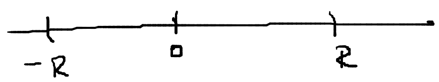
Теорема: $$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}} \overset{(*)}{=} \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}} \overset{(*)}{=} \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

① $a_n \downarrow$, $\lim\limits_{n\to\infty} a_n = 0$, $\sum\limits_{n=1}^{\infty} a_n$ дивергира

Одредити обл. конв. $\sum\limits_{n=1}^{\infty} \frac{a_n}{e^{a_n}} x^n$. $\quad I = [-R, R)$, $R = ?$

обл. конв. $\quad -R, R \in I$?

$x_0 = 0$



$a_n \downarrow$, $\lim\limits_{n\to+\infty} a_n = 0 \geq 0 \Rightarrow a_n > 0$

$$\frac{\frac{a_n}{e^{a_n}}}{\frac{a_{n+1}}{e^{a_{n+1}}}} = \underbrace{\frac{a_n}{a_{n+1}}}_{>1} \cdot \underbrace{e^{\overbrace{a_{n+1}-a_n}^{\leq 0}}}_{<1} \longrightarrow ?$$

$$\frac{a_n^{1/n}}{e^{a_n/n}} = \frac{e^{1/n \ln a_n}}{e^{a_n/n}} = e^{\frac{1}{n}(\ln a_n - a_n)} \longrightarrow ?$$

$\frac{1}{n} \to 0$, $\ln a_n \to -\infty$, $a_n \to 0$ $\quad (n \to +\infty)$

Т. о израчунавању радијуса не даје ништа.

$\frac{a_n}{e^{a_n}} \cdot x^n \sim a_n x^n \quad , n \to +\infty \quad \Rightarrow \quad x = 1 \qquad \frac{a_n}{e^{a_n}} \sim a_n$

$e^{a_n} \to \underset{\substack{\| \\ 1}}{e^0} \quad , \quad a_n \to 0$ $\qquad \overset{\text{II п. крит.}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{a_n}{e^{a_n}} \cdot 1^n$ дивергира

$\sum_{n=1}^{\infty} \underbrace{\frac{a_n}{e^{a_n}}} \cdot (-1)^n$

$\frac{a_n}{e^{a_n}} > 0 \quad , \quad \frac{a_n}{\underset{\substack{\downarrow \\ 1}}{e^{a_n}}} \underset{\substack{\downarrow \\ a_n \to 0}}{\longrightarrow} 0 \quad , \quad \frac{a_n}{e^{a_n}} \downarrow \; ?$

$f(x) = \frac{x}{e^x} \quad \uparrow \; ?$ $\qquad f(x) = x \cdot e^{-x}$

$f'(x) = e^{-x} - x \cdot e^{-x} = e^{-x}(1-x) \underset{\substack{\downarrow \\ x<1}}{>} 0$

$a_n \downarrow$

$a_n \geq a_{n+1}$

$f(a_n) \geq f(a_{n+1})$

$\Rightarrow \quad f \uparrow \qquad x < 1$

$a_n \to 0 \quad \Rightarrow \quad \forall n \geq n_0 \quad a_n < 1$

$\Rightarrow \quad f(a_n) \geq f(a_{n+1})$

$\frac{a_n}{e^{a_n}} \geq \frac{a_{n+1}}{e^{a_{n+1}}} \quad , \quad n \geq n_0$

Лајбниц $\Rightarrow \sum_{n=1}^{\infty} \frac{a_n}{e^{a_n}} (-1)^n$ конв.

Лема $\Rightarrow \quad R = 1 \quad$ и $\quad I = \underline{[-1,1)}$

(2) Одр. обл. конв. $\sum_{n=1}^{\infty} \sin(\sqrt{n+1} - \sqrt{n}) \, (x - \underset{\substack{\| \\ x_0}}{2})^n$.

$x_0 = 2$

$I = \underset{\uparrow}{} \; 2-R \, , \, R+2 \; \underset{\uparrow}{} \; \to$ облик обл. конв. $\quad R = ?$

видећемо какве су заграде

$a_n = \sin(\sqrt{n+1} - \sqrt{n}) = \sin\left((\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right) = \sin \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \sin \frac{1}{\sqrt{n+1} + \sqrt{n}} \underset{\substack{\downarrow \\ n \geq n_0}}{>} 0$

$\sqrt[n]{a_n} = a_n^{1/n} = e^{1/n \cdot \ln a_n} \sim e^{\frac{1}{n} \ln \frac{1}{2\sqrt{n}}} \sim e^{-\frac{\ln \sqrt{n}}{n}} \sim 1$, $\quad -\frac{\ln\sqrt{n}}{n} \to 0$

$a_n = \sin \underbrace{\frac{1}{\sqrt{n+1} + \sqrt{n}}}_{\downarrow \, 0} \sim \frac{1}{\sqrt{n+1} + \sqrt{n}} \sim \frac{1}{2\sqrt{n}}$

$R = 1$

$\Rightarrow \quad I = \underset{\uparrow}{(}\, 2-1, 2+1 \,\underset{\uparrow}{)} = (1,3)$

$x = 1:$ $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)\cdot(1-2)^n = \sum_{n=1}^{\infty} (-1)^n \overbrace{\underbrace{\sin\frac{1}{\sqrt{n+1}+\sqrt{n}}}_{>0}}^{a_n}$

$\sin\frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \sin\frac{1}{\sqrt{n+2}+\sqrt{n+1}}$

$a_n \downarrow$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to+\infty} \sin\frac{1}{\sqrt{n+2}+\sqrt{n+1}}$ $\Bigg\}\overset{\text{Лајбниц}}{\Longrightarrow} \displaystyle\sum_{n=1}^{\infty} a_n(-1)^n$ конв.

$\Rightarrow 1 \in I$

$x=3$ $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)\cdot\underbrace{(3-2)^n}_{=1} = \sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)$

$\sin\frac{1}{\sqrt{n+1}+\sqrt{n}} \sim \frac{1}{\sqrt{n+1}+\sqrt{n}} \sim \frac{1}{2\sqrt{n}}, \; n\to+\infty$

$\sum \frac{1}{\sqrt{n}}$ дивергира $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{\sqrt{n+1}+\sqrt{n}}$ див. $\Rightarrow 3 \notin I$

$\boxed{I = [1,3)}$.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{2^n n^3}$ одр. обл. конв.