* Лајбницово правило:

$\sum_{n=1}^{\infty} (-1)^n c_n$ : $c_n \geq 0$, $\downarrow$, $\lim_{n\to\infty} c_n = 0$ $\Rightarrow$ $\sum (-1)^n c_n$ конвергира

① $\underbrace{\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^2+2n} - n}{n^{4/3}}}_{a_n}$

⌈ $\sum a_n$, $a_n \in \mathbb{R}$ апс. конв. ако конв. $\sum |a_n|$

$\sum a_n$ апс. конв. $\Rightarrow$ $\sum a_n$ условно
$\not\Leftarrow$

$\sum \frac{(-1)^n}{n} \to$ конв условно и не конв. апс. јер $\sum \frac{1}{n}$ див. ⌋

$\sum |a_n|$ конв?

$\sum |a_n| = \sum \frac{\sqrt{n^2+2n} - n}{n^{4/3}}$

$|a_n| = \frac{\sqrt{n^2+2n} - n}{n^{4/3}} = \frac{n\sqrt{1+\frac{2}{n}} - n}{n^{4/3}} \sim \frac{n(\cancel{1}+\frac{1}{n}) - \cancel{n}}{n^{4/3}}$ ($\sim 1+\frac{1}{2}\cdot\frac{2}{n}$)

$\sim \frac{1}{n^{4/3}}$, $n \to +\infty$

$4/3 > 1$ $\Rightarrow$ $\sum \frac{1}{n^{4/3}}$ конв. $\xRightarrow[\text{крит.}]{\text{II поређ.}}$ $\sum |a_n|$ конв.

$\Rightarrow \sum a_n = \sum (-1)^n |a_n|$ конв. апсолутно.

② $\sum_{n=2}^{\infty} \frac{(-1)^n}{\underbrace{\sqrt{n}+(-1)^n}_{a_n}}$, $a_n = (-1)^n c_n$, $c_n = \frac{1}{\sqrt{n}+(-1)^n}$

$\sum |a_n|$, $|a_n| = \frac{1}{\sqrt{n}+(-1)^n} \sim \frac{1}{\sqrt{n}}$, $n \to +\infty$, $\sum \frac{1}{\sqrt{n}}$ диверг.

$\Rightarrow \sum |a_n|$ див.

$c_n \geq 0$, $\lim_{n\to\infty} c_n = 0$ $\checkmark$

$c_n \downarrow ?$

$$\frac{1}{\sqrt{n}+(-1)^n} \geq \frac{1}{\sqrt{n+1}+(-1)^{n+1}} \ ? \iff \underline{\sqrt{n+1}+(-1)^{n+1} \geq \sqrt{n}+(-1)^n \ ?}$$

$n$ парно

$$\sqrt{n+1} - 1 \overset{?}{\geq} \sqrt{n}+1$$

$$\sqrt{n+1} \geq \sqrt{n}+2 \quad \uparrow^2 \implies n+1 \geq n+4+4\sqrt{n} \ ↯$$

$c_n$ не опаѓа! не може Лајбниц!

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$$

$$\frac{(-1)^n}{\sqrt{n}+(-1)^n} + \frac{(-1)^{n+1}}{\sqrt{n+1}+(-1)^{n+1}} = \frac{(-1)^n\sqrt{n+1} - 1 + (-1)^{n+1}\sqrt{n} - 1}{(\sqrt{n}+(-1)^n)(\sqrt{n+1}+(-1)^{n+1})}$$

$(\sqrt{n}\cdot(1+\frac{1}{2}\cdot\frac{1}{n})-\sqrt{n})$

$$= \frac{(-1)^n(\sqrt{n+1}-\sqrt{n}) - 2}{\underbrace{(\quad)(\quad)}_{\sim n}} \sim \frac{(-1)^n \cdot \frac{1}{2\sqrt{n}} - 2}{n} \checkmark$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty}\sum_{k=1}^{n-1} a_k$$

$\Downarrow$

$\lim a_n = 0$

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n} \cdot \frac{\sqrt{n}-(-1)^n}{\sqrt{n}-(-1)^n} = \sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{n} - 1}{n-1} \overset{(*)}{=}$$

$$= \underbrace{\sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{n}}{n-1}}_{\text{конв.}} - \underbrace{\sum_{n=2}^{\infty}\frac{1}{n-1}}_{\text{див.}} \implies \sum a_n \text{ див.}$$

Лајбниц:

$c_n = \frac{\sqrt{n}}{n-1}$, $c_n \downarrow$, $c_n \geq 0$, $\lim_{n\to\infty} c_n = 0$

$$\frac{\sqrt{n}}{n-1} \overset{?}{\geq} \frac{\sqrt{n+1}}{n}$$

$$n\sqrt{n} \geq \sqrt{n+1}\cdot(n-1) \quad \uparrow^2$$

$$n^3 \geq (n+1)^2(n-1)^2 = (n+1)(n^2-2n+1)$$

$$0 \geq n^2 - 2n^2 + n + 1 - 2n = \underline{-n^2 - n + 1}$$

(3) $\sum\limits_{n=1}^{\infty} \underbrace{\frac{(-1)^n}{(n^{2p}+2)^3}}_{a_n}$ , $p \in \mathbb{R}$

1° $p>0$

$\sum |a_n| = \sum \frac{1}{(n^{2p}+2)^3}$ , $|a_n| = \frac{1}{(n^{2p}+2)^3} \sim \frac{1}{n^{6p}}$

$\sum \frac{1}{n^{6p}}$ конв. акко $6p>1$

$\Rightarrow \sum |a_n|$ конв акко $6p>1$. $(p>\frac{1}{6})$

2° $p<0$ , $n^{2p} \to 0$ , $n \to +\infty$

$|a_n| \sim \frac{1}{8}$ , $\lim\limits_{n\to\infty} |a_n| = \frac{1}{8} \neq 0$

$\Downarrow$

$\lim\limits_{n\to\infty} a_n \neq 0$ (неће постојати)

3° $p=0 \Rightarrow a_n = \frac{1}{27} \not\to 0$

$p \le 0 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ дивергира

4° $0<p<\frac{1}{6}$ $\sum a_n$ условно конв ?

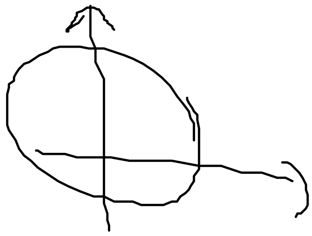
$a_n = (-1)^n \underbrace{\frac{1}{(n^{2p}+2)^3}}_{c_n}$

$c_n \ge 0$ , $\lim\limits_{n\to\infty} c_n = 0$

$c_n \downarrow$ ? $\frac{1}{(n^{2p}+2)^3} \downarrow$ $(n^{2p} \uparrow$ за $p>0)$

Лајбниц $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ условно конв за $0<p<\frac{1}{6}$.

(4) $\sum_{n=1}^{\infty} \underbrace{\sin\left(\pi\sqrt{n^2+a^2}\right)}_{a_n}$



$$a_n=\sin\left(\left(\pi\sqrt{n^2+a^2}-n\pi\right)+n\pi\right)=$$

$$=\sin\left(\pi\sqrt{n^2+a^2}-n\pi\right)\underbrace{\cos n\pi}_{(-1)^n}+\cancel{\sin n\pi}^{\,0}\cdot\cos\left(\pi\sqrt{n^2+a^2}-n\pi\right)$$

$$=(-1)^n\sin\left(\pi\sqrt{n^2+a^2}-n\pi\right)$$

$$\left(\sqrt{n^2+a^2}-n\right)\cdot\frac{\sqrt{n^2+a^2}+n}{\sqrt{n^2+a^2}+n}=\frac{n^2+a^2-n^2}{\sqrt{n^2+a^2}+n}=\frac{a^2}{\sqrt{n^2+a^2}+n}$$

$$a_n=(-1)^n\sin\frac{a^2\pi}{\sqrt{n^2+a^2}+n}$$

$\frac{a^2\pi}{\sqrt{n^2+a^2}+n}\geq 0$, $\downarrow$ по $n$, $\lim\limits_{n\to+\infty}\frac{a^2\pi}{\sqrt{n^2+a^2}+n}=0$

$\sin\uparrow$ на $\left(0,\frac{\pi}{2}\right)$ и непр.

$\Rightarrow$ $\sin\frac{a^2\pi}{\sqrt{n^2+a^2}+n}\geq 0$, $n\geq n_0$

$\sin$ —||— $\downarrow$

$$\lim_{n\to+\infty}\sin\frac{a^2\pi}{\sqrt{n^2+a^2}+n}=0.$$

Лајбниц $\Rightarrow$ $\sum\underbrace{(-1)^n\sin\frac{a^2\pi}{\sqrt{n^2+a^2}+n}}_{a_n}$ конв. (условно проверено)

$\sin x\sim x,\ x\to 0$

$$|a_n|=\sin\frac{a^2\pi}{\sqrt{n^2+a^2}+n}\overset{?}{\sim}\frac{a^2\pi}{\sqrt{n^2+a^2}+n}\sim\frac{a^2\pi}{2n},\quad\sum\frac{1}{n}\text{ див.}$$

$\sum |a_n|$ дивергира.

$\sum_{n=1}^{\infty} a_n b_n$ конв?

* Абелово правило

А1. $\sum_{n=1}^{\infty} a_n$ конвергира

А2. $b_n$ ограничен и монотон

$\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ конв.

* Дирихлеово правило

Д1. $S_n = \sum_{k=1}^{n} a_k$ је огран. низ

Д2. $b_n$ монотон и $\lim_{n\to\infty} b_n = 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ конв.

Лема: $S_n = \sum_{k=1}^{n} \sin k\alpha$, низ $S_n$ је ограничен.

1° $\alpha = m\pi$, $m \in \mathbb{Z}$ $\quad \sin \underbrace{k \cdot m\pi}_{\in \mathbb{Z}} = 0 \quad \Rightarrow S_n = 0$

2° $\alpha \neq m\pi$, $m \in \mathbb{Z}$

$$S_n = (\sin 1\cdot\alpha + \sin 2\cdot\alpha + \dots \sin n\cdot\alpha) \cdot 2\sin\frac{\alpha}{2}$$

$$2\sin\frac{\alpha}{2} S_n = 2\sin\frac{\alpha}{2}\cdot\sin\alpha + 2\sin\frac{\alpha}{2}\cdot\sin 2\alpha + \dots + 2\sin\frac{\alpha}{2}\sin n\alpha$$

$$= \cos\frac{\alpha}{2} - \cancel{\cos\frac{3\alpha}{2}} + \cancel{\cos\frac{3\alpha}{2}} - \cancel{\cos\frac{5\alpha}{2}} + \dots + \cancel{\cos\left(n-\frac{1}{2}\right)\alpha} - \cos\left(n+\frac{1}{2}\right)\alpha$$

$$= \cos\frac{\alpha}{2} - \cos\left(n+\frac{1}{2}\right)\alpha$$

$$|S_n| = \left|\frac{\cos\frac{\alpha}{2} - \cos\left(n+\frac{1}{2}\right)\alpha}{2\sin\frac{\alpha}{2}}\right| \leq \frac{2}{2\left|\sin\frac{\alpha}{2}\right|} < +\infty$$

$\alpha \neq m\pi$

$\sum_{n=1}^{\infty} \cos n\alpha$ има огран. парц. суме

① $\sum_{n=1}^{\infty} \frac{\cos 2n}{n^{\alpha}}$, $\alpha > 0$

$a_n = \cos 2n$, $b_n = \frac{1}{n^{\alpha}}$

Д2. $b_n = \frac{1}{n^{\alpha}} \downarrow$, $\lim_{n\to\infty} \frac{1}{n^{\alpha}} = 0$ ✓

$n^{\alpha} \uparrow$, $\alpha > 0$

Д1. $a_n = \cos 2n$, $\underline{\underline{\alpha = 2}}$

$$S_n = \sum_{k=1}^{n} \cos 2k = (\cos 2 + \cos 4 + \dots + \cos 2n) \,/\, 2\sin 1$$

$$\frac{\alpha}{2}$$

$$2\sin 1 \cdot S_n = 2\sin 1\cos 2 + 2\sin 1 \cos 4 + \dots 2\sin 1 \cos 2n$$

$$= \underbrace{\cancel{\sin 3}} - \underline{\sin 1} + \cancel{\sin 5} - \cancel{\sin 3} + \cancel{\sin 7} - \sin 5$$

$$+ \dots + \underbrace{\sin(2n+1)} - \cancel{\sin(2n-1)}$$

$$= \sin(2n+1) - \sin 1$$

$$|S_n| = \left| \frac{\sin(2n+1) - \sin 1}{2\sin 1} \right| \le \frac{1}{\sin 1}$$

$\Rightarrow$ $S_n$ je огран низ. ✓

Дирихле $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{\cos 2n}{n^\alpha}$ конв. (условно)

$$\left| \frac{\cos 2n}{n^\alpha} \right| \leq \frac{1}{n^\alpha}$$

$\sum \frac{1}{n^\alpha}$ конв ако је $\alpha > 1$

I поредб. крит. $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{\cos 2n}{n^\alpha}$ конв. апс. за $\alpha > 1$

$$|a_n| = \frac{|\cos 2n|^{\in[0,1]}}{n^\alpha} \geq \frac{\cos^2 2n}{n^\alpha} = \frac{1+\cos 4n}{2n^\alpha}$$

$$\sum |a_n| \overset{*}{\geq} \underbrace{\frac{1}{2}\sum \frac{1}{n^\alpha}}_{\substack{\text{див.} \\ \text{ако } 0<\alpha\leq 1}} + \underbrace{\frac{1}{2}\sum \frac{\cos 4n}{n^\alpha}}_{\substack{\text{преко Дирихлеовог правила} \\ \text{конв.}}}$$

I поредб. крит. $\Rightarrow$ $\sum |a_n|$ див.

② $\sum_{n=1}^{\infty} \frac{\sin n \ln n}{n}$

1° $\frac{\ln n}{n} \downarrow$ $\left( f(x) = \frac{\ln x}{x}, \quad f'(x) = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2} \right.$

$\left. f' < 0, \quad x \geq 3 \Rightarrow f(n) = \frac{\ln n}{n} \downarrow n \geq 3 \right)$

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

А1. $\sum_{n=1}^{\infty} \sin n$ огран. парц. суме.

Дирихле $\Rightarrow$ $\sum_{n=1}^{\infty} \sin n \frac{\ln n}{n}$ условно конв.

апс. конв. $\to$ за венбу испитати

③ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\frac{\pi}{n}}{\ln^2 n}$

апс. конв. за венбу.

А1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln^2 n} \xrightarrow{\text{Лајбниц}}$ конвергира

✓

$\frac{1}{\ln^2 n} \downarrow$ , $\lim_{n\to\infty} \frac{1}{\ln^2 n} = 0$ , $\frac{1}{\ln^2 n} > 0$

А2. $\cos\frac{\pi}{n} \uparrow$

✓ $\frac{\pi}{n} \downarrow$ + $\cos \downarrow$ на $(0, \frac{\pi}{2})$ , $\lim_{n\to\infty} \frac{\pi}{n} = 0$

$\lim_{n\to\infty} \cos\frac{\pi}{n} = 1$

$\Rightarrow \cos\frac{\pi}{n}$ огран.

Абел $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\frac{\pi}{n}}{\ln^2 n}$ условно конв.

④ $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n}{\sqrt{n}}}_{a_n} \underbrace{\int_0^n e^{-x^2}dx}_{b_n}$

А1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, $\frac{1}{\sqrt{n}} > 0$, $\downarrow$, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Лейбниц $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ конв.

А2. $b_n \uparrow$, $e^{-x^2} > 0$ $\Rightarrow \int_0^n e^{-x^2}dx < \int_0^{n+1} e^{-x^2}dx$

$$\int_0^{n+1} e^{-x^2}dx = \int_0^n + \underbrace{\int_n^{n+1} \underset{0}{\underset{\vee}{e^{-x^2}}} dx}_{\vee\, 0}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \int_0^n e^{-x^2}dx = \int_0^{+\infty} e^{-x^2}dx < +\infty\ ?$$

$x > 1$ $\quad e^{-x^2} < e^{-x}$

$$\underbrace{\int_1^{+\infty} e^{-x^2}dx}_{\text{конв.}} < \int_1^{+\infty} e^{-x}dx \leq 1 < +\infty$$

$$\int_0^{+\infty} e^{-x^2}dx \text{ конв.}$$

Абел

$\Rightarrow$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \underbrace{\int_0^n e^{-x^2}dx}_{b_n}$ конв. условно.

$$|a_n| = \frac{\int_0^n e^{-x^2}dx}{\sqrt{n}} \sim \frac{\ell = \int_0^{+\infty} e^{-x^2}dx}{\sqrt{n}}, \quad n \to +\infty$$

$\sum \frac{1}{\sqrt{n}}$ диверґира $\Rightarrow \sum |a_n|$ диверґ.

за вежбу:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+5}}, \quad \sum_{n=1}^{\infty} (-1)^n n \cdot \left(e^{\frac{1}{n^2}} - 1\right)^p$$

$$\sum_{n=1}^{\infty} (-1)^n \operatorname{arctg} n \cdot \operatorname{arctg} \frac{1}{n}, \quad \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^n}{n^{n+1}}$$

5 $\sum_{n=0}^{\infty} \binom{\alpha}{n}, \ \alpha \in \mathbb{R}$ конв?

$$\binom{\alpha}{n} = \frac{\alpha \cdot (\alpha-1) \cdots (\alpha-n+1)}{n!}$$

1° $\alpha \in \mathbb{N} \cup \{0\}$ $\quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-\alpha)\cdots(\alpha-n+1)}{n!} = 0$

$n > \alpha$

$$\Rightarrow \sum_{n=0}^{\infty} \binom{\alpha}{n} = \sum_{n=0}^{\alpha} \binom{\alpha}{n} \quad \text{конв.}$$

$2^\circ$ $\alpha \notin \mathbb{N} \cup \{0\}$

За $n > \alpha$ $a_n$ и $a_{n+1}$ су различитог знака

$a_n = \binom{\alpha}{n}$ $\underbrace{a_{n+1} = a_n \cdot \frac{\alpha - n}{n+1}}$

$\underset{n>\alpha}{?}$ $a_n \to 0$ ?

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|\alpha - n|}{|n+1|} = \frac{n-\alpha}{n+1} = 1 - \underbrace{\frac{\alpha+1}{n+1}}$$

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n-\alpha} = 1 + \frac{\alpha+1}{n} + \frac{C}{n^2} \quad - \frac{\alpha+1}{n} + \frac{const}{n^2}$$

$|a_n| \nearrow \Leftrightarrow -\left(\frac{\alpha+1}{n+1}\right) \geq 0 \Leftrightarrow \alpha \leq -1 \Rightarrow |a_n| \nearrow$

$\alpha \leq -1$

$\Rightarrow \lim\limits_{n\to\infty} |a_n| \neq 0 \Rightarrow \lim\limits_{n\to\infty} a_n \neq 0 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ дивергира

$1^\circ$ $\alpha \leq -1 \Rightarrow \sum a_n$ див.

$2^\circ$ $\alpha + 1 > 1$, $\alpha > 0 \overset{\text{Гаус}}{\Rightarrow} \sum |a_n|$ конв.

$\alpha > 0 \Rightarrow \sum a_n$ апс. конв.

$\alpha \leq 0 \Rightarrow \sum |a_n|$ див. (и даље може условно да конв.)

$3^\circ$ $-1 < \alpha < 0$ : $\sum_{n=1}^{\infty} \binom{\alpha}{n}$

$a_n = (-1)^n \cdot |a_n|$

$|a_n| \downarrow$ , $\lim_{n\to\infty} |a_n| = 0$ ?

$|a_n| \downarrow$

$$\left| \frac{a_{n+1}}{a_n} \right| = 1 - \frac{\overset{>0}{\alpha+1}}{n+1} < 1$$

$\Rightarrow$ $|a_n| \downarrow$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{(0-\alpha)(1-\alpha)(2-\alpha)\cdots(n-1-\alpha)}{n!}$$

$\underbrace{\lim_{n\to+\infty} \ln|a_n|}_{} \overset{?}{=} -\infty \Rightarrow \lim_{n\to\infty} |a_n| = 0$

$$\ln|a_n| = \ln(0-\alpha) + \ln(1-\alpha) + \cdots + \ln(n-1-\alpha)$$
$$- (\ln 1 + \ln 2 + \cdots + \ln n)$$

$$= \ln \frac{0-\alpha}{1} + \ln \frac{1-\alpha}{2} + \cdots + \ln\left(\frac{n-1-\alpha}{n}\right)$$

$$= \ln\left(1 - \frac{\alpha+1}{1}\right) + \ln\left(1 - \frac{\alpha+1}{2}\right) + \cdots + \ln\left(1 - \frac{1+\alpha}{n}\right)$$

$< \, ? \longrightarrow -\infty$

$$\ln\left(1-\frac{\alpha+1}{n}\right) < -\frac{\alpha+1}{n+1}$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \ldots$$

$$\ln(1-x) < -x$$

$\ln(1-x)+x$

$f'(x) = (\ln(1-x)+x)' = \frac{1}{1-x}\cdot(-1)+1 < 0$

$f \downarrow$

$f(0)=0$

$f(x)<0$, $x>0$

$$x = \frac{\alpha+1}{n}$$

$$\ln\left(1-\frac{\alpha+1}{n}\right) < -\frac{\alpha+1}{n}$$

$$\ln|a_n| < \sum_{k=1}^{n} -\frac{\alpha+1}{k} = -\underset{\substack{\vee\\0}}{(\alpha+1)}\underbrace{\sum_{k=1}^{n}\frac{1}{k}}_{\to+\infty} \to -\infty$$

$$\lim_{n\to\infty}\ln|a_n| = -\infty \Rightarrow \lim_{n\to\infty}|a_n| = 0$$

Лейбниц. $\Rightarrow$ $\sum\binom{\alpha}{n}$ конв. условно $-1<\alpha<0$.