· Кумерово правило: $\sum_{n=1}^{\infty} a_n$, $a_n > 0$ конв?

$\sum_{n=1}^{\infty} \frac{1}{c_n}$ дивергира, $c_n \geq 0$

$K_n = c_n \frac{a_n}{a_{n+1}} - c_{n+1}$

1° $\exists \delta > 0$ $\exists n_0 \in \mathbb{N}$ $K_n \geq \delta > 0$, $n \geq n_0$ $\Rightarrow \sum_{n=1}^{\infty} a_n$ конвергираћи

$\exists n_0 \in \mathbb{N}$ $K_n \leq 0$, $n \geq n_0$ $\Rightarrow \sum_{n=1}^{\infty} a_n$ дивергира

2° $\lim_{n\to\infty} K_n = K$: $K > 0 \Rightarrow \sum a_n$ конв.

$K < 0 \Rightarrow \sum a_n$ див.

$c_n = n$, $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \frac{1}{c_n} = \sum_{n=1}^{\infty} \frac{1}{n}$ дивергира

$K_n = n \frac{a_n}{a_{n+1}} - (n+1)$

$\exists \delta > 0$ $\exists n_0 \in \mathbb{N}$ $n \frac{a_n}{a_{n+1}} - (n+1) \geq \delta$, $n \geq n_0 \Rightarrow \sum a_n$ конв

$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq \underbrace{\delta + 1}_{r > 1}$, $n \geq n_0 \Rightarrow \sum a_n$ конв.

$n\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1$, $n \geq n_0 \Rightarrow \sum a_n$ дивергира

Рабеово правило

$c_n = n \ln n$, $\sum \frac{1}{c_n}$ дивергира

$K_n = n \ln n \cdot \frac{a_n}{a_{n+1}} - (n+1)\ln(n+1)$

$K_n \geq \delta > 0$

$-n\ln n + n \ln n$

$n \ln n \frac{a_n}{a_{n+1}} - (n+1)\ln(n+1) \geq \delta$

$n \ln n \left(\frac{a_n}{a_{n+1}} - 1\right) + \underbrace{n\ln n - (n+1)\ln(n+1)} \geq \delta$

$\frac{a_n}{a_{n+1}} - 1 \geq \frac{\delta + (n+1)\ln(n+1) - n\ln n - \ln n + \ln n}{n \ln n} = \frac{\delta + (n+1)\ln\left(1+\frac{1}{n}\right)}{n \ln n} + \frac{1}{n}$

$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \underbrace{\frac{\delta}{n \ln n}}_{o\left(\frac{1}{n}\right)} + \underbrace{\left(1+\frac{1}{n}\right) \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}}_{\text{ово треба бити мало}}{}^{\sim \frac{1}{n}} \sim o\left(\frac{1}{n}\right) \Rightarrow \sum a_n$ конв.

Гаусово правило:

$$\frac{a_n}{a_{n+1}} = \lambda + \frac{\mu}{n} + \frac{\theta_n}{n^\alpha}, \quad \alpha > 1, \ \theta_n \text{ ограничен низ}$$

а) $\lambda > 1 \Rightarrow \sum a_n$ конвергира

б) $\lambda < 1 \Rightarrow \sum a_n$ дивергира

в) $\lambda = 1$: $\mu > 1$ $\sum a_n$ конвергира

$\mu \leq 1$ $\sum a_n$ дивергира

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① $\sum\limits_{n=1}^{\infty} \underbrace{\frac{\sqrt{n!}}{(2+\sqrt{1})\cdot(2+\sqrt{2})\cdots(2+\sqrt{n})}}_{a_n}$ конв ?

$$\frac{a_n}{a_{n+1}} = \frac{\dfrac{\cancel{\sqrt{n!}}}{\cancel{(2+\sqrt{1})(2+\sqrt{2})\cdots(2+\sqrt{n})}}}{\dfrac{\sqrt{(n+1)!}}{\cancel{(2+\sqrt{1})\cdots(2+\sqrt{n})}(2+\sqrt{n+1})}} = \frac{2+\sqrt{n+1}}{\sqrt{n+1}} = \frac{2}{\sqrt{n+1}} + 1$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{2n}{\sqrt{n+1}} \to +\infty \xRightarrow[\text{правило}]{\text{Рабеово}} \sum_{n=1}^{\infty} a_n \text{ конвергира}$$

② $\sum\limits_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}$, $\quad (2n-1)!! = 1\cdot3\cdot5\cdots(2n-1)$

$(2n)!! = 2\cdot4\cdot6\cdot8\cdots(2n)$

$$\frac{a_n}{a_{n+1}} = \frac{\dfrac{\cancel{(2n-1)!!}}{\cancel{(2n)!!}\cdot(2n+1)}}{\dfrac{\cancel{(2n+1)!!}}{\cancel{(2n+2)!!}\cdot(2n+3)}} = \frac{(2n+2)(2n+3)}{(2n+1)^2} \geq 1$$

$$\frac{a_{n+1}}{a_n} \leq 1 \longrightarrow \text{Даламбер не даје ништа}$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{(2n+2)(2n+3) - (2n+1)^2}{(2n+1)^2}\right) =$$

$$= n \cdot \frac{\cancel{4n^2} + 10n + 6 - \cancel{4n^2} - 4n - 1}{(2n+1)^2} = n \cdot \frac{6n+5}{(2n+1)^2} = \frac{6n^2+5n}{4n^2+4n+1} = 1 + \underbrace{\frac{2n^2+n-1}{4n^2+4n+1}}_{(*)\ \downarrow\ n\to\infty\ \ 2/4 \ \ \vee\ 0}$$

у $(*)$ $\varepsilon = \frac{1}{4}$

$$\Rightarrow \exists n_0 \ \forall n \geq n_0 \ \ n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq 1 + \frac{1}{4} > 1 \xRightarrow{\text{Рабе}} \sum_{n=1}^{\infty} a_n \text{ конв.}$$

(3) $\sum_{n=1}^{\infty} \frac{(2n)!!}{(2n-1)^{n+e}} \cdot e^{n}$ $\to$ за вежбу

(4) $\sum_{n=1}^{\infty} \underbrace{\left(\frac{p(p+1)\cdots(p+n-1)}{q(q+1)\cdots(q+n-1)}\right)^{\alpha}}_{a_n}$ конв ? , $p,q>0$, $\alpha \in \mathbb{R}\setminus\{0\}$

$$\frac{a_n}{a_{n+1}} = \left(\frac{\dfrac{p(p+1)\cdots\cancel{(p+n-1)}}{q\cancel{(q+1)\cdots}(q+n-1)}}{\dfrac{p(p+1)\cdots\cancel{(p+n-1)}(p+n)}{q\cancel{(q+1)\cdots}(q+n-1)(q+n)}}\right)^{\alpha} = \left(\frac{q+n}{p+n}\right)^{\alpha} =$$

$$= \frac{\left(1+\frac{q}{n}\right)^{\alpha}}{\left(1+p/n\right)^{\alpha}} = \left(1+\underbrace{\frac{q}{n}}_{\to 0}\right)^{\alpha}\cdot\left(1+\underbrace{\frac{p}{n}}_{\to 0}\right)^{-\alpha}$$

$$= \left(1+\alpha\frac{q}{n}+\frac{\alpha(\alpha-1)}{2}\cdot\frac{q^2}{n^2}+o\left(\frac{1}{n^2}\right)\right)\left(1-\alpha\frac{p}{n}+\frac{(-\alpha)(-\alpha-1)}{2}\frac{p^2}{n^2}+o\left(\frac{1}{n^2}\right)\right)$$

$$(1+x)^{\beta} = 1+\underset{\beta}{\binom{\beta}{1}}x+\underset{\frac{\beta(\beta-1)}{2}}{\binom{\beta}{2}}x^2+o(x^2),\ x\to 0$$

$$\frac{c}{n^2}+o\left(\frac{1}{n^2}\right) = \overset{\to 0}{\alpha(n)}\cdot\frac{1}{n^2}$$

$$= \underset{\lambda}{1}+\alpha\frac{q-p}{n}\underbrace{-\alpha^2\frac{pq}{n^2}+\frac{\alpha(\alpha-1)}{2}\frac{q^2}{n^2}+\frac{\alpha(\alpha+1)}{2}\frac{p^2}{n^2}+o\left(\frac{1}{n^2}\right)}_{\frac{\theta_n}{n^2}}$$

, θ_n ограничено

(није константно због $o\left(\frac{1}{n^2}\right)$
али како $\frac{o\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} \to 0$
θ_n ће бити огран.)

$\theta_n = c+\alpha(n) \to c$

$\lambda = 1$

$\mu = \alpha\cdot(q-p)$ >1 $\Rightarrow \sum a_n$ конв.

≤ 1 $\Rightarrow \sum a_n$ дивергира

* $f \in C^1[0,1]$ $\quad \int_0^1 f(x)\,dx = 0 \quad \overset{?}{\Rightarrow} \quad \forall \alpha \in [0,1] \quad \left| \int_0^\alpha f(x)\,dx \right| \le \frac{1}{8} \max_{x\in[0,1]} |f'(x)|$

$$F(\alpha) = \int_0^\alpha f(x)\,dx \quad , \; F \in C^2(0,1) \quad , \; F(0) = F(1) = 0$$

$$F'(\alpha) = f(\alpha) \quad , \; F''(\alpha) = f'(\alpha)$$

$$|F(\alpha)| \overset{?}{\le} \frac{1}{8} \max_{x\in[0,1]} |F''(x)|$$

$a \in (0,1)$

$$F(x) = F(a) + \underbrace{\frac{F'(a)}{1!}\cdot(x-a)}_{} + \underbrace{\frac{F''(\xi_x)}{2!}(x-a)^2}_{\text{Лагранжев остатак}} \quad , \; \xi_x \in (x,a)$$

$f \not\equiv 0$

$F(0) = F(1) = 0 \quad \Rightarrow \quad \exists\, a \in (0,1) \; : \; |F(a)| = \max_{x\in[0,1]} |F(x)|$

Вајерштрас на $|F|$

$\left.\begin{array}{l} a \to \text{локални екстремум за } F \\ F \in C^1 \end{array}\right\} \Rightarrow F'(a) = 0$

$$F(x) = F(a) + \frac{F''(\xi_x)}{2}\cdot(x-a)^2 \quad , \; x \in [0,1]$$

$\xi_x \in (a,x)$

$x = 0$

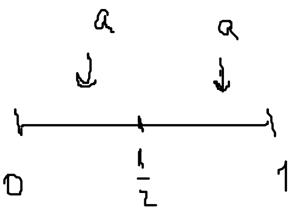
$$0 = F(0) = F(a) + \frac{\overset{=f'}{F''}(\xi_0)}{2} a^2 \quad , \; \xi_0 \in (0,a)$$

$x = 1$

$$0 = F(1) = F(a) + \frac{F''(\xi_1)}{2}(1-a)^2 \quad , \; \xi_1 \in (a,1)$$

$$F(a) = -\frac{f'(\xi_0)}{2} a^2$$

$$F(a) = -\frac{f'(\xi_1)}{2}(1-a)^2$$



$$|F(a)| = \frac{|f'(\xi_0)|}{2}\cdot a^2 \le \frac{\max|f'|}{2}\cdot a^2$$

$$= \frac{|f'(\xi_1)|}{2}(1-a)^2 \le \frac{\max|f'|}{2}\cdot(1-a)^2$$

$|F(x)| \leq |F(a)| \leq \frac{\max|f'|}{2} \cdot \min\{a^2, (1-a)^2\} \leq \frac{\max|f'|}{8}$

$\downarrow$

због одабира тачке a

$a \in (0,1)$

$a^2 \leq \frac{1}{4}$ или $(1-a)^2 \leq \frac{1}{4}$

Редови са произвољним члановима

$\sum_{n=1}^{\infty} a_n$, $a_n \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$

$\sum_{n=1}^{\infty} |a_n|$ конв. $\Rightarrow \sum_{n=1}^{\infty} a_n$ апсолутно конвергира (онда $\sum a_n$ конв.)

иначе ако $\sum_{n=1}^{\infty} |a_n|$ див., а $\sum_{n=1}^{\infty} a_n$ конв., онда

$\sum_{n=1}^{\infty} a_n$ условно конв.

$\sum_{n=1}^{\infty} (-1)^n c_n \rightarrow$ алтернирајући редови, $c_n \geq 0$

* Лајбницово правило:

$c_n \geq 0$, $\lim_{n\to\infty} c_n = 0$, $c_n \downarrow \Rightarrow \sum_{n=1}^{\infty} (-1)^n c_n$ конвергира

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ конвергира, $c_n = \frac{1}{n} \geq 0$, $\downarrow$, $\lim_{n\to\infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ дивергира

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ условно конвергира

① $$\sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{1+(-1)^n n}{n^2}}_{\text{није увек позитивно}} = \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^k \frac{1+(-1)^k k}{k^2}$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{(-1)^k}{k^2} + \sum_{k=1}^{n} \frac{\overbrace{(-1)^k\cdot(-1)^k}^{=1}}{k}\right)$$

$$\overset{(*)}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^k}{k^2} + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}}_{\text{конвергира по Лајбницу}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{дивергира } (+\infty)}$$

(*) уколико постоји у $\overline{\mathbb{R}}$

$c_n = \frac{1}{n^2} \geq 0, \downarrow, \lim_{n\to\infty} c_n = 0$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1+(-1)^n n}{n^2}$ дивергира (као збир конв и див. реда)

$$\sum \underbrace{\frac{4\cdot 7 \cdots (3n+1)}{2\cdot 6 \cdots (4n-2)}}_{a_n}$$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{4\cdot 7\cdots(3n+1)}{2\cdot 6\cdots(4n-2)}}{\frac{4\cdots(3n+4)}{(4n+4)}} = \frac{4n+2}{3n+4}$$

$$\frac{a_{n+1}}{a_n} = \frac{3n+4}{4n+4} \leq L < 1 \sim \frac{3}{4} \quad a_n \downarrow$$

$a_1 = a$

$a_n \approx \left(\frac{3}{4}\right)^n \cdot a \longrightarrow 0$