$\cdot \sum_{n=1}^{\infty} a_n \overset{(*)}{:=} \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} a_n}_{s_n}$ (*) уколико десна стр. постоји једнакост важи
ако не постоји или је $\pm\infty$, $\sum_{n=1}^{\infty} a_n$ дивергира
иначе $\sum_{n=1}^{\infty} a_n$ конв.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ конв. акко } p>1}$$

· Поређени критеријум: $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$

I: $0 \le a_n \le b_n$: $\sum_{n=1}^{\infty} b_n$ конв. $\Rightarrow \sum_{n=1}^{\infty} a_n$ конв.

$\sum a_n$ див. $\Rightarrow \sum b_n$ див. $\left(\sum = \sum_{n=1}^{\infty}\right)$

II: $a_n \ge 0$, $b_n > 0$, $n \ge n_0$: $\lim_{n\to\infty} \frac{a_n}{b_n} = C$

1° $C < \infty$: $\sum b_n$ конв. $\Rightarrow \sum a_n$ конв.

2° $C > 0$: $\sum a_n$ конв $\Rightarrow \sum b_n$ конв.

3° $0 < C < \infty$: $\sum a_n$ конв. $\Leftrightarrow \sum b_n$ конв.

① $\sum_{n=1}^{\infty} \underbrace{\frac{\ln(2n+1) - \ln 2}{\sqrt[3]{n^4+12}} \cdot \left(e^{\frac{1}{\sqrt{n}}} - 1\right)}_{a_n \downarrow 0}$ конв?

$a_n \to 0$ и $a_n > 0$

$$\frac{\ln(2n+1) - \ln 2}{\sqrt[3]{n^4+12}} = \frac{\ln \frac{2n+1}{2}}{\sqrt[3]{n^4+12}} \sim \frac{\ln n}{n^{4/3}}, \quad n\to\infty$$

$$e^{\frac{1}{\sqrt{n}}} - 1 \sim 1 + \frac{1}{\sqrt{n}} - 1 = \frac{1}{\sqrt{n}}, \quad n\to\infty \qquad \left(f(n) \sim g(n),\ n\to\infty \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1\right)$$

$$a_n \sim \frac{\ln n}{n^{4/3}} \cdot \frac{1}{\underset{n^{1/2}}{\sqrt{n}}} = \frac{\ln n}{n^{11/6}}, \quad n\to\infty, \quad \sum_{n=1}^{\infty} \frac{\ln n}{n^{11/6}} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{11/6}(\ln n)^{-1}} \text{ конв јер } \frac{11}{6} > 1}_{\text{задатак са прошлог часа}}$$

II поређ. крит. $\Rightarrow \sum_{n=1}^{\infty} a_n$ конв.

② Нека $\underbrace{\sum_{n=1}^{\infty} a_n}_{a_n \ge 0}$ конв. Да ли онда конв. $\sum_{n=1}^{\infty} a_n^2$?

$a_n \to 0$, $n\to\infty$ јер $\sum_{n=1}^{\infty} a_n$ конв.

$\exists n_0 \in \mathbb{N}$ $n \ge n_0$ $a_n \le 1$ $\Rightarrow$ $0 \le a_n^2 \le a_n$

I поређ. крит. $\Rightarrow \sum_{n=1}^{\infty} a_n^2$ конв.

③ Ако је $\lim\limits_{n\to\infty} na_n = a > 0$, да ли $\sum\limits_{n=1}^{\infty} a_n$ конв ?

$$\lim_{n\to\infty} na_n = a \iff na_n \sim a \,,\, n\to\infty \iff a_n \sim \frac{a}{n} \,,\, n\to\infty$$

$$\underset{\underset{0}{\vee}}{a}\sum_{n=1}^{\infty}\frac{1}{n} \text{ дивергира } \overset{\text{II пореð. крит.}}{\Longrightarrow} \sum_{n=1}^{\infty} a_n \text{ дивергира.}$$

④ $$\sum_{n=1}^{\infty} \underbrace{\left(1-\cos\frac{\pi}{n^3}\right)^{p}\left(\sqrt[3]{n^3+1}-n\right)^{q}}_{a_n} \quad , p,q\in\mathbb{R}$$

$$\cos x \sim 1-\frac{x^2}{2} \,,\, x\to 0$$

$$\cos\frac{\pi}{n^3} \sim 1-\frac{\pi^2}{2n^6} \,,\, n\to\infty$$

$$1-\cos\frac{\pi}{n^3} \sim \frac{\pi^2}{2n^6} \,,\, n\to\infty$$

$$\left(1-\cos\frac{\pi}{n^3}\right)^{p} \sim \frac{\pi^{2p}}{2^p n^{6p}} \,,\, n\to\infty$$

$$\sqrt[3]{n^3\left(1+\frac{1}{n^3}\right)}-n = n\sqrt[3]{1+\frac{1}{n^3}}-n \sim n\left(1+\frac{1}{n^3}\cdot\frac{1}{3}\right)-n = \frac{1}{3n^2} \,,\, n\to\infty$$

$$\left(\text{—}||\text{—}\right)^{q} \sim \frac{1}{3^q n^{2q}} \,,\, n\to\infty$$

$$a_n \sim \frac{\pi^{2p}}{2^p\cdot 3^q}\cdot\frac{1}{n^{6p+2q}} \,,\, n\to\infty$$

$$\underline{\text{II}} \text{ поредб. крит.} \Longrightarrow \sum a_n \text{ конв. акко } \sum\frac{1}{n^{6p+2q}} \text{ конв. акко } 6p+2q>1$$

⑤ $$\sum_{n=2}^{\infty} \underbrace{\ln^{p}\frac{1}{\cos\frac{\pi}{n}}\left(\sqrt{n+1}-\sqrt{n}\right)^{q}\ln\frac{n+1}{n-1}}_{a_n}$$

$$\ln\frac{1}{\cos\frac{\pi}{n}} \sim \ln\frac{1}{1-\frac{\pi^2}{2n^2}} = \ln\left(\left(1-\frac{\pi^2}{2n^2}\right)^{-1}\right) \sim \ln\left(1-1\cdot\left(-\frac{\pi^2}{2n^2}\right)\right) = \ln\left(1+\frac{\pi^2}{2n^2}\right) \sim \frac{\pi^2}{2n^2} \,,\, n\to\infty$$

$$\sqrt{n+1}-\sqrt{n} = \sqrt{n}\left(1+\frac{1}{n}\right)^{1/2}-\sqrt{n} \sim \sqrt{n}\left(1+\frac{1}{2n}\right)-\sqrt{n} = \frac{1}{2\sqrt{n}} \,,\, n\to\infty$$

$$\ln\frac{n+1}{n-1} = \ln\left(1+\frac{2}{n-1}\right) \sim \frac{2}{n-1} \sim \frac{2}{n} \,,\, n\to\infty$$

$$a_n \sim \left(\frac{\pi^2}{2n^2}\right)^{p}\cdot\left(\frac{1}{2\sqrt{n}}\right)^{q}\cdot\frac{2}{n} = \frac{\pi^{2p}\cdot 2}{2^p\cdot 2^q}\cdot\frac{1}{n^{2p+q/2+1}} \,,\, n\to\infty$$

$$\sum a_n \text{ конв.} \overset{\text{II пореð. крит.}}{\iff} \sum\frac{1}{n^{2p+q/2+1}} \text{ конв.} \iff 2p+q/2+1>1 \iff 2p+q/2>0 \,.$$

⑥ $$\sum_{n=1}^{\infty} \underbrace{\left(e-\left(1+\frac{1}{n}\right)^{n}\right)^{p}}_{a_n}$$

$$e-\left(1+\frac{1}{n}\right)^{n} = e-e^{n\ln\left(1+\frac{1}{n}\right)} \sim e-e^{n\left(\frac{1}{n}-\frac{1}{2n^2}\right)} = e-e^{1-\frac{1}{2n}} = e\left(1-e^{-\frac{1}{2n}}\right)$$

$$\sim e\left(1-\left(1-\frac{1}{2n}\right)\right) \sim e\cdot\frac{1}{2n} \,,\, n\to\infty$$

$$a_n \sim \frac{e^p}{2^p n^p}, \; n \to \infty$$

$\sum a_n$ конв. акко $\sum \frac{1}{n^p}$ конв акко $p > 1$.

за ~~вежбу~~ вежбу: $\sum_{n=1}^{\infty} n^{\alpha}\left(\frac{1}{\sqrt{n}} - \sqrt{\ln\left(1+\frac{1}{n}\right)}\right), \; \alpha \in \mathbb{R}$

$\sum a_n, \; a_n > 0$

* Даламберово правило:

1° $\exists n_0 \in \mathbb{N} \quad q \in \mathbb{R} \qquad \frac{a_{n+1}}{a_n} \leq q < 1, \; n \geq n_0 \Rightarrow \sum a_n$ конвергира

$\frac{a_{n+1}}{a_n} \geq 1, \; n \geq n_0 \Rightarrow \sum a_n$ дивергира

2° $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$: $l < 1 \Rightarrow \sum a_n$ конвергирати

$l > 1 \Rightarrow \sum a_n$ дивергира

$l = 1$ не знамо

* Кошијево правило:

1° $\exists n_0 \in \mathbb{N} \quad q \in \mathbb{R} \qquad \sqrt[n]{a_n} \leq q < 1, \; n \geq n_0 \Rightarrow \sum a_n$ конвергира

$\sqrt[n]{a_n} \geq 1, \; n \geq n_0 \Rightarrow \sum a_n$ дивергира

2° $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = l$: $l < 1 \Rightarrow \sum a_n$ конвергира

$l > 1 \Rightarrow \sum a_n$ дивергира

$l = 1$ не знамо

① $\sum_{n=1}^{\infty} \underbrace{(\sqrt{5} - \sqrt[3]{5})(\sqrt{5} - \sqrt[5]{5}) \cdot \ldots \cdot (\sqrt{5} - \sqrt[2n+1]{5})}_{a_n}$ конв?

$$a_{n+1} = (\sqrt{5} - \sqrt[3]{5})(\sqrt{5} - \sqrt[5]{5}) \cdots (\sqrt{5} - \sqrt[2n+1]{5})(\sqrt{5} - \sqrt[2n+3]{5})$$

$$\frac{a_{n+1}}{a_n} = \sqrt{5} - \sqrt[2n+3]{5} \quad / \lim_{n\to\infty}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \sqrt{5} - \underbrace{\lim_{n\to\infty} \sqrt[2n+3]{5}}_{\to 1} = \sqrt{5} - 1 > 1 \overset{\text{Даламбер}}{\Longrightarrow} \sum a_n \text{ дивергира}$$

② $$\sum_{n=1}^{\infty} \underbrace{\left(\frac{n-1}{n+1}\right)^{n(n-1)}}_{a_n}$$

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n-1)}} = \left(\frac{n-1}{n+1}\right)^{n-1} = e^{(n-1)\ln\left(\frac{n-1}{n+1}\right)} = e^{(n-1)\ln\left(1-\frac{2}{n+1}\right)} = e^{(n-1)\left(\left(-\frac{2}{n+1}\right)+o\left(\frac{1}{n}\right)\right)}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{(n-1)\left(-\frac{2}{n+1}+o\left(\frac{1}{n}\right)\right)} = e^{-2} = \frac{1}{e^2} < 1$$

Кошијево правило $\Rightarrow$ $\sum a_n$ конвергира

③ $$\sum_{n=1}^{\infty} \underbrace{\frac{2^n}{n!}}_{a_n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{\cancel{2^n}}{\cancel{n!}}} = \frac{2}{n+1} \longrightarrow 0, \ n\to\infty < 1 \quad \overset{\text{Даламбер}}{\Rightarrow} \quad \sum a_n \text{ конв.}$$

④ $$\sum_{n=1}^{\infty} \underbrace{nx \prod_{k=1}^{n} \frac{\sin^2(ka)}{1+x^2+\cos^2(ka)}}_{a_n}, \ x>0 \qquad \prod_{k=1}^{n} b_k = b_1\cdot b_2\cdot b_3 \cdots b_n$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)x \prod_{k=1}^{n+1} \frac{\sin^2(ka)}{1+x^2+\cos^2(ka)}}{nx \cancel{\prod_{k=1}^{n} \frac{\sin^2(ka)}{1+x^2+\cos^2(ka)}}} = \frac{n+1}{n}\cdot\frac{\sin^2((n+1)a)}{1+x^2+\cos^2((n+1)a)}$$

$$\frac{\sin^2((n+1)a)}{1+x^2+\cos^2((n+1)a)} \le \frac{1}{1+x^2} < 1$$

$$\frac{n+1}{n} = 1+\frac{1}{n}$$

$$\frac{a_{n+1}}{a_n} \le \left(1+\frac{1}{n}\right)\cdot\frac{1}{1+x^2} = \frac{1}{1+x^2} + \frac{1}{n}\cdot\frac{1}{1+x^2} \longrightarrow \frac{1}{1+x^2} < 1, \ n\to\infty \qquad x\neq 0$$

$$\exists n_0\in\mathbb{N} \quad \frac{a_{n+1}}{a_n} < \frac{1}{1+x^2} + \varepsilon < 1$$

$$\varepsilon = \frac{1-\frac{1}{1+x^2}}{2}$$

Даламбер $\Rightarrow$ $\sum a_n$ конвергира

⑤ $$\sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n} \text{ конв?}$$

$$\sqrt[n]{\frac{n^{n+1/n}}{(n+1/n)^n}} = \frac{n^{1+1/n^2}}{n+\frac{1}{n}} = \frac{n^{1/n^2}}{1+\frac{1}{n^2}} = \frac{e^{\frac{\ln n}{n^2}}}{1+\frac{1}{n^2}} = \frac{1+\frac{\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right)}{1+\frac{1}{n^2}} \ge 1, \ n\ge n_0$$

Коши $\Rightarrow \sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$ дивергира

За вежбу: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$, $\sum_{n=1}^{\infty} 2^n \left(\frac{2n-1}{2n}\right)^{n^2}$, $\sum_{n=1}^{\infty} \frac{4\cdot 7\cdot 10\cdots(3n+1)}{2\cdot 6\cdot 10\cdots(4n-2)}$