* $\int_0^{+\infty} \frac{|\sin x|}{x}\,dx \overset{(*)}{=} \lim_{\beta\to+\infty}\int_0^{\beta} \frac{|\sin x|}{x}\,dx$

0 — отклоњив синг.

$\underbrace{\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}}\int_0^{n\pi}\frac{|\sin x|}{x}\,dx \text{ не постоји у } \mathbb{R}}$ $\Rightarrow$ $\int_0^{+\infty}\frac{|\sin x|}{x}\,dx$ дивергира

$$\int_0^{n\pi}\frac{|\sin x|}{x}\,dx = \int_0^{\pi}\dots + \int_{\pi}^{2\pi}\dots + \dots + \int_{(n-1)\pi}^{n\pi}\dots \geq \frac{2}{\pi}\underbrace{\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)}_{\downarrow\ +\infty} \Big/ \lim_{n\to\infty}$$

$$\int_{(k-1)\pi}^{k\pi}\underbrace{\frac{|\sin x|}{x}}_{\geq 0}\,dx \geq \frac{1}{k\pi}\underbrace{\int_{(k-1)\pi}^{k\pi}|\sin x|\,dx}_{=2} = \frac{2}{k\pi}$$

$\frac{1}{x}\geq\frac{1}{k\pi}$

$\frac{|\sin x|}{x}\geq\frac{|\sin x|}{k\pi}$

$\Rightarrow \lim_{n\to\infty}\int_0^{n\pi}\frac{|\sin x|}{x}\,dx = +\infty \Rightarrow \int_0^{+\infty}\frac{|\sin x|}{x}\,dx$ дивергира.

Абелов крит. $\int_a^{b} f(x)g(x)\,dx$ Дирихлеов крит

А1. $f\in C[a,b)$, $\int_a^b f(x)\,dx$ конв. $\Rightarrow$ Д1. $f\in C[a,b)$, $F(\beta)=\int_a^{\beta} f(x)\,dx$ огран по β

А2. $g\in C^1[a,b)$, монотона, огран. $\Leftarrow$ Д2. $g\in C^1[a,b)$, монотона, $\lim_{x\to b} g(x)=0$.

$\Rightarrow \int_a^b f(x)g(x)\,dx$ конв.

за вежбу не конв. апсолутно.

(1) $\int_2^{+\infty}\underbrace{\frac{\cos x}{\ln^2 x}}_{f(x)}\,\underbrace{\operatorname{arctg} x}_{g(x)}\,dx$

Абелов крит:

А1. $\frac{\cos x}{\ln^2 x}\in C[2,+\infty)$

$\int_2^{+\infty}\frac{\cos x}{\ln^2 x}\,dx$ конв:

Д1 ✓ $\cos x\in C[2,+\infty)$ $F(\beta)=\int_2^{\beta}\cos x\,dx = \sin\beta-\sin 2$

$|F(\beta)|\leq 2 \Rightarrow F$ огран

Д.2. $\frac{1}{\ln^2 x} \in C^1[2,+\infty)$, $\frac{1}{\ln^2 x} \downarrow$, $\lim\limits_{x\to+\infty} \frac{1}{\ln^2 x} = 0$ ✓

$\Rightarrow$ Дирихлеов крит. $\int\limits_2^{+\infty} \frac{\cos x}{\ln^2 x}\,dx$ конв.

А2. $g(x) = \operatorname{arctg} x \in C^1[2,+\infty)$, $\operatorname{arctg} x \uparrow$, $\lim\limits_{x\to+\infty} \operatorname{arctg} x = \frac{\pi}{2}$ $\Rightarrow \operatorname{arctg} x$ огран на $[2,+\infty)$

$\Rightarrow$ Абелов крит. $\int\limits_2^{+\infty} \frac{\cos x}{\ln^2 x} \operatorname{arctg} x\,dx$ конв.

② $\int\limits_0^{+\infty} \frac{\cos^2 x}{\sqrt{x}}\,dx$ конв. ?

$\cos^2 x = \frac{1+\cos 2x}{2}$ $\to$ [примитивна ф-ја од $\cos^2 x$ није огран]

$$\int\limits_0^{+\infty} \frac{\cos^2 x}{\sqrt{x}}\,dx = \underbrace{\int\limits_0^1 \frac{\cos^2 x}{\sqrt{x}}\,dx}_{} + \underbrace{\int\limits_1^{+\infty} \frac{\cos^2 x}{\sqrt{x}}\,dx}_{\text{див.}} \Rightarrow \int\limits_0^{+\infty} \frac{\cos^2 x}{\sqrt{x}}\,dx \text{ див.}$$

$\frac{\cos^2 x}{\sqrt{x}} \sim \frac{1}{\sqrt{x}}$, $x \to 0^+$

$\int\limits_0^1 \frac{dx}{\sqrt{x}}$ конв. $\xRightarrow{\text{II поред. крит.}}$ $\int\limits_0^1 \frac{\cos x}{\sqrt{x}}\,dx$ конв.

$$\underbrace{\int\limits_1^{+\infty} \frac{\cos^2 x}{\sqrt{x}}\,dx}_{\text{див.}} \overset{(*)}{=} \underbrace{\underbrace{\int\limits_1^{+\infty} \frac{1}{2\sqrt{x}}\,dx}_{\text{дивергира}} + \underbrace{\int\limits_1^{+\infty} \frac{\cos 2x}{2\sqrt{x}}\,dx}_{\text{конв}}}_{\text{див.}}$$

$\int\limits_1^{+\infty} \frac{\cos 2x}{\sqrt{x}}\,dx$:

Д1. $\cos 2x \in C[1,+\infty)$, $F(\beta) = \int\limits_1^{\beta} \cos 2x\,dx = \frac{1}{2}(\sin 2\beta - \sin 2)$
$|F(\beta)| \le 1$ ✓

Д2. $\frac{1}{\sqrt{x}} \in C^1[1,+\infty)$, $\frac{1}{\sqrt{x}} \downarrow$, $\lim\limits_{x\to+\infty} \frac{1}{\sqrt{x}} = 0$ ✓

Дирихлеов крит. $\Rightarrow$ $\int\limits_1^{+\infty} \frac{\cos 2x}{\sqrt{x}}\,dx$ конв.

$$\int\limits_a^b f(x)\,dx := \int\limits_a^c f(x)\,dx + \int\limits_c^b f(x)\,dx$$

по деф. лева стран конв. акко конв. оба интеграла са десне стр.
(довољно је да један интеграл див. да би дивергирао $\int\limits_a^b f(x)\,dx$

$$\int_{a}^{b} f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \underbrace{\lim_{\alpha\to a^+}\int_\alpha^c f(x)\,dx + \lim_{\beta\to b^-}\int_c^\beta f(x)\,dx}_{\alpha \text{ и } \beta \text{ су независни}}$$

$(-\infty + \infty) \Rightarrow$ дивергира

$$\int_a^b f(x)+g(x)\,dx \overset{(*)}{=} \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(*) → ово важи само ако не добијемо недефинисану ситуацију

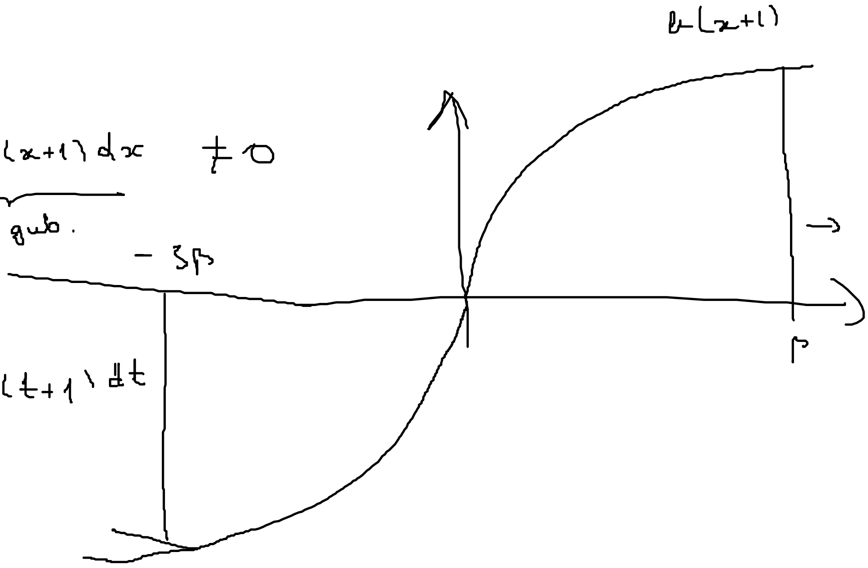
$$\lim_{\beta\to b^-}\int_a^\beta f(x)+g(x)\,dx = \lim_{\beta\to b^-}\left(\int_a^\beta f(x)\,dx + \int_a^\beta g(x)\,dx\right) \overset{(*)}{=} \lim_{\beta\to b^-}\int_a^\beta f(x)\,dx + \lim_{\beta\to b^-}\int_a^\beta g(x)\,dx$$

када можемо да протегнемо lim кроз збир

$$f(x) = \begin{cases} \ln(x+1), & x \geq 0 \\ -\ln(-x+1), & x \leq 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} -\ln(-x+1)\,dx + \underbrace{\int_0^{+\infty} \ln(x+1)\,dx}_{\text{див.}} \neq 0$$

↓ дивергира



$$-\int_{-\infty}^{0} \ln(-x+1)\,dx = \left[t = -x \right] = -\int_0^{+\infty} \ln(t+1)\,dt$$

③ $\displaystyle\int_2^{+\infty} \frac{dx}{x^\alpha \ln^\beta x}$, $\alpha, \beta \in \mathbb{R}$ конв?

$$\int_2^{+\infty} \frac{dx}{x^\alpha \ln^\beta x} = \left[\begin{array}{l} t = \ln x, \quad x = e^t \\ dt = \frac{dx}{x} \\ dx = e^t dt \end{array} \right] = \int_{\ln 2}^{+\infty} \frac{e^t\,dt}{e^{\alpha t} t^\beta} = \int_{\ln 2}^{+\infty} \frac{dt}{e^{(\alpha-1)t} t^\beta}$$

1° $\alpha-1>0$, $\alpha>1$

Γ $e^{(\alpha-1)t} > t^{\square}$ → било шта

$\frac{1}{e^{(\alpha-1)t}} < \frac{1}{t^{\square}}$ → било шта

$\frac{1}{e^{(\alpha-1)t}\cdot t^{\beta}} < \frac{1}{t^2}$ ⌟

$$\lim_{t\to+\infty} \frac{\frac{1}{e^{(\alpha-1)t}\cdot t^{\beta}}}{\frac{1}{t^2}} = \lim_{t\to+\infty} \frac{t^{2-\beta}}{e^{\overset{(\alpha-1)}{\underset{>0}{}}t}} = 0$$

II поредб. крит.

$$\int_{\ln 2}^{+\infty} \frac{dt}{t^2} \text{ конв.} \implies \int_{\ln 2}^{+\infty} \frac{dt}{e^{(\alpha-1)t}\cdot t^{\beta}} \text{ конв.}$$ → (независно од β!)

2° $\alpha-1<0$

Γ $e^{(\overset{>0}{1-\alpha})t} > t^{\square}$ → било шта

$\frac{1}{e^{(\alpha-1)t}\cdot t^{\beta}} > t^{\square}$ → било шта ⌟

$$\lim_{t\to+\infty} \frac{\frac{1}{e^{(\alpha-1)t}\cdot t^{\beta}}}{\frac{1}{t}} = \lim_{t\to+\infty} \frac{e^{(1-\alpha)t}}{t^{\beta-1}} = +\infty$$

$$\int_{\ln 2}^{+\infty} \frac{dt}{t} \text{ дивергира } \overset{\text{II поредб.}}{\underset{\text{крит.}}{\implies}} \int_{\ln 2}^{+\infty} \frac{dt}{e^{(\alpha-1)t}\cdot t^{\beta}} \text{ дивергира}$$ (независно од β)

3° $\alpha=1$

$$\int_{\ln(2)}^{+\infty} \frac{dt}{e^{(\alpha-1)t}\, t^{\beta}} = \int_{\ln(2)}^{+\infty} \frac{dt}{t^{\beta}} \text{ конв.} \iff \underline{\underline{\beta>1}}$$

$$\int_{2}^{+\infty} \frac{dx}{x^{\alpha}\ln^{\beta} x} \text{ конв.} \iff \underset{\beta\in\mathbb{R}}{\alpha>1} \text{ или } (\alpha=1 \text{ и } \beta>1).$$

За вежбу, испитати конв. $\int_{1}^{+\infty} \frac{\sin x}{x}\, \mathrm{arctg}\, x\, dx$, $\int_{0}^{+\infty} \frac{\cos ax}{1+x^4}\, dx$.

условно и апс.

* Нека $\int_0^{+\infty} f(x)\,dx$ конвергира.

а) Да ли онда мора да постоји $\lim\limits_{x\to+\infty} f(x)$? НЕ!

$f(x)=\cos(x^2)$

$x_n^2 = 2n\pi$, $x_n=\sqrt{2n\pi} \xrightarrow[n\to+\infty]{} +\infty$, $f(x_n)=\cos 2n\pi = 1$ $\Rightarrow$ $\lim\limits_{x\to+\infty} f(x)$ не постоји

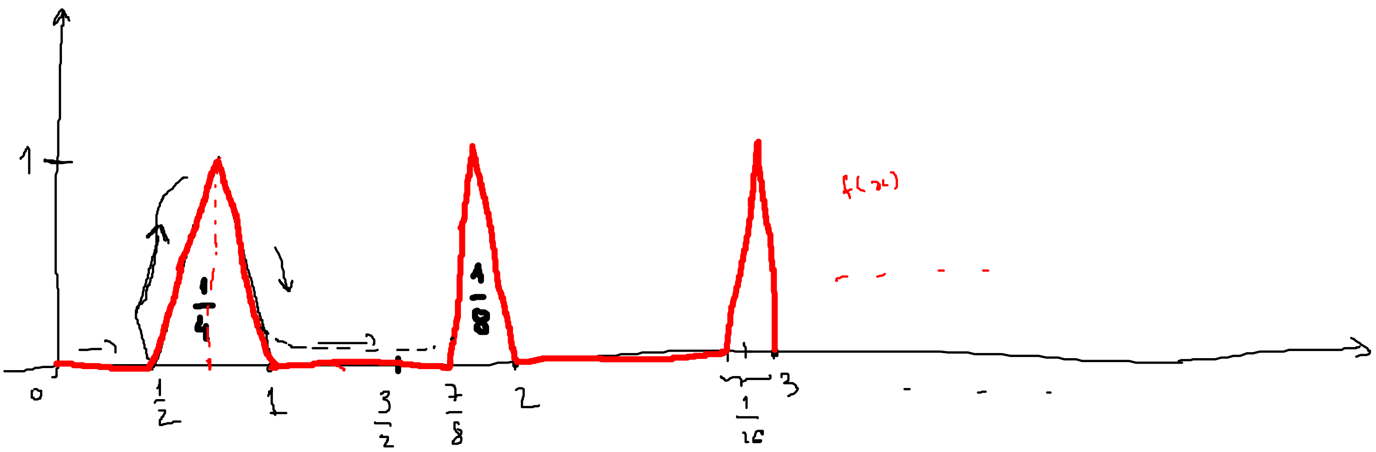
$y_n^2 = 2n\pi+\frac{\pi}{2}$, $y_n = \sqrt{2n\pi+\frac{\pi}{2}} \xrightarrow[n\to+\infty]{} +\infty$, $f(y_n)=\cos(2n\pi+\frac{\pi}{2})=0$

$$\int_0^{+\infty}\cos(x^2)\,dx = \left| \begin{array}{l} t=x^2 \\ x=\sqrt{t} \\ dx=\frac{dt}{2\sqrt{t}} \end{array}\right| = \int_0^{+\infty}\frac{\cos t}{2\sqrt{t}}\,dt \to \text{конв. по Дирихлеу}$$

б) Ако је $f(x)\geq 0$, да ли онда мора постојати $\lim\limits_{x\to+\infty} f(x)$? НЕ!

$f(x)>0$ сличан пример

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2^k} = \lim_{n\to\infty}\frac{1}{2}\,\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1$$



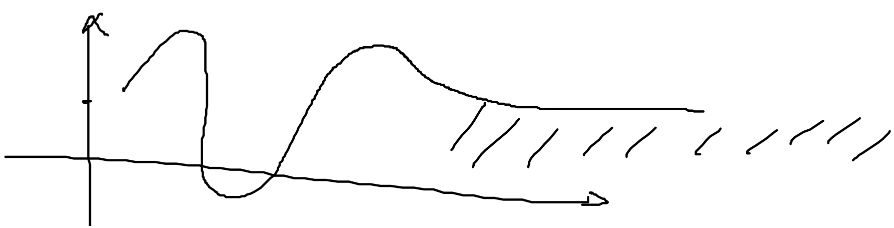
$$f(x)=\begin{cases} 2^{n+1}x - 2^{n+1}n+2, & x\in\left(n-\frac{1}{2^n}, n-\frac{1}{2^{n+1}}\right) \\ -2^{n+1}x+2^{n+1}n, & x\in\left(n-\frac{1}{2^{n+1}}, n\right] \\ 0, & \text{иначе} \end{cases}, \quad n\in\mathbb{N}$$

$\lim\limits_{x\to\infty} f(x)$ не постоји

али $\int_0^{+\infty} f(x)\,dx = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{2^{k+1}} = \frac{1}{2}$

\- - - -

в) Ако $\lim\limits_{x\to+\infty} f(x)$ постоји онда $\lim\limits_{x\to+\infty} f(x) = 0$!



У супротном ако $\lim\limits_{x\to+\infty} f(x) = A > 0$.

$\forall \varepsilon > 0 \quad \exists M > 0 \quad x > M \Rightarrow |f(x) - A| < \varepsilon \Rightarrow f(x) \in (A-\varepsilon, A+\varepsilon)$

$\varepsilon = \frac{A}{2} > 0, \quad x > M \Rightarrow f(x) > A - \varepsilon = \frac{A}{2} > 0$

$$\int_0^{+\infty} f(x)\,dx = \int_0^M f(x)\,dx + \int_M^{+\infty} \underbrace{f(x)}_{\geq \frac{A}{2}}\,dx \geq \int_0^M f(x)\,dx + \frac{A}{2}\underbrace{\int_M^{+\infty} dx}_{+\infty}$$

$\int_0^{+\infty} f(x)\,dx$ дивергира!