Поредбени критеријуми

I : $0 \le f(x) \le g(x)$, $x \in [a,b)$ $\Rightarrow 0 \le \int_a^{(b)} f(x)\,dx \le \int_a^{(b)} g(x)\,dx$

$\int_a^b g(x)\,dx$ конв. $\Rightarrow \int_a^b f(x)\,dx$ конв.

$\int_a^b f(x)\,dx$ див. $\Rightarrow \int_a^b g(x)\,dx$ див.

II : $0 < g(x)$, $x \in [a,b)$, $0 \le f(x)$, $x \in [a,b)$, $\lim_{x\to b^-} \frac{f(x)}{g(x)} = C \in [0,+\infty]$

1° $C < +\infty$ $\int_a^{(b)} g(x)\,dx$ конв. $\Rightarrow \int_a^{(b)} f(x)\,dx$ конв.

$\int_a^{(b)} f(x)\,dx$ див. $\Rightarrow \int_a^{(b)} g(x)\,dx$ див.

2° $C > 0$: $\int_a^{(b)} f(x)\,dx$ конв. $\Rightarrow \int_a^{(b)} g(x)\,dx$ конв.

$\int_a^b g(x)\,dx$ див. $\Rightarrow \int_a^b f(x)\,dx$ див.

3° $0 < C < +\infty$ $\int_a^{(b)} f(x)\,dx$ конв. $\Leftrightarrow \int_a^{(b)} g(x)\,dx$ конв.

$\int_0^c \frac{dx}{x^p}$ конв. $\Leftrightarrow$ $p < 1$

$\int_c^{+\infty} \frac{dx}{x^p}$ конв. $\Leftrightarrow$ $p > 1$

$\int_a^{(b)} f(x)\,dx$ конв. $\Leftrightarrow \int_c^{(b)} f(x)\,dx$ конв. $c \in (a,b)$

(1) $\int_{(a)}^{(b)} \frac{dx}{\sqrt{(x-a)(b-x)}}$ конв ? ако да, израчунати. $b > a$

$$= \int_{(a)}^{\frac{a+b}{2}} \frac{dx}{\sqrt{(x-a)(b-x)}} + \int_{\frac{a+b}{2}}^{(b)} \frac{dx}{\sqrt{(x-a)(b-x)}}$$

$$\int_a^{\frac{a+b}{2}} \frac{dx}{\sqrt{(x-a)(b-x)}} = \left[\begin{array}{l} x-a=t \\ dx=dt \end{array}\right] = \int_{(0)}^{\frac{b-a}{2}} \frac{dt}{\sqrt{t(b-a-t)}}$$

$$\frac{1}{\sqrt{t\underbrace{(b-a-t)}_{\downarrow \, b-a}}} \sim \frac{1}{\sqrt{b-a}}\frac{1}{t^{1/2}}, \quad t\to 0^+ \qquad \left(\lim_{t\to 0^+} \frac{\frac{1}{\sqrt{b-a}\cdot t^{1/2}}}{\frac{1}{\sqrt{t(b-a-t)}}} = 1 \right)$$

II поредб. крит.

$$\int_{0}^{\frac{b-a}{2}} \frac{dt}{\sqrt{b-a}\cdot t^{1/2}} \text{ конв. } \left(p=\frac{1}{2}\right) \Rightarrow \int_0^{\frac{b-a}{2}} \frac{dt}{\sqrt{t(b-a-t)}} \text{ конв.}$$

$$\Rightarrow \int_a^{\frac{a+b}{2}} \frac{dx}{\sqrt{(x-a)(b-x)}} \text{ конв.}$$

$$\int_{\frac{a+b}{2}}^{b} \frac{dx}{\sqrt{(x-a)(b-x)}} = \begin{bmatrix} b-x=t \\ -dx=dt \end{bmatrix} = \int_{\frac{b-a}{2}}^{0} \frac{-dt}{\sqrt{(b-a-t)t}} = \int_0^{\frac{b-a}{2}} \frac{dt}{\sqrt{t(b-a-t)}} \text{ конв.}$$

$$\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} \text{ конв.}$$

$$\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} =$$

$\Big[\, x = a\cos^2 t + b\sin^2 t, \quad t \in (0, \frac{\pi}{2})$

$x(0) = a$

$x(\frac{\pi}{2}) = b$

$x - a = a\cos^2 t + b\sin^2 t - a(\cos^2 t + \sin^2 t)$

$= (b-a)\sin^2 t$

$b - x = (b-a)\cos^2 t$

$dx = -2a\cos t\sin t + 2b\sin t\cos t$

$= 2(b-a)\cos t\sin t\,dt\,\Big]$

$$= \int_0^{\pi/2} \frac{2(b-a)\cos t\sin t\,dt}{\sqrt{(b-a)\sin^2 t\,(b-a)\cos^2 t}} = \int_0^{\pi/2} \frac{2\cancel{(b-a)}\cancel{\cos t}\sin t}{\cancel{(b-a)}\cancel{\cos t}\sin t}\,dt = 2\cdot\frac{\pi}{2} = \pi.$$

$\left.\begin{matrix}\sin t>0 \\ \cos t>0\end{matrix}\right\}$ зер $t\in(0,\pi/2)$

② $p, q \in \mathbb{R}$ $\quad \displaystyle\int_0^{\pi/2} \frac{dx}{\sin^p x \cos^q x}$ конв ?

1° $p, q \le 0$ $\quad \displaystyle\int_0^{\pi/2} \frac{dx}{\sin^p x \cos^q x}$ → својствен интеграл
⇒ конв.

$$\int_0^{\pi/2} \frac{dx}{\sin^p x \cos^q x} = \int_0^{\pi/4} \frac{dx}{\sin^p x \cos^q x} + \underbrace{\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^p x \cos^q x}}_{t = \frac{\pi}{2} - x}$$

$\sin x = \cos t$
$\cos x = \sin t$

$$= \int_0^{\pi/4} \frac{dx}{\sin^p x \cos^q x} + \int_0^{\pi/4} \frac{dt}{\cos^p x \sin^q x}$$

$\displaystyle\int_0^{\pi/4} \frac{dx}{\sin^p x \cos^q x}$: $\quad \dfrac{1}{\sin^p x \underbrace{\cos^q x}_{\to 1}} \sim \dfrac{1}{x^p}$, $\quad x \to 0^+$

$\sin x \sim x$

$\displaystyle\int_0^{\pi/4} \frac{dx}{x^p}$ конв. ⇔ $p < 1$

II поредб. крит. ⇒ $\displaystyle\int_0^{\pi/4} \frac{dx}{\sin^p x \cos^q x}$ конв. ⇔ $p < 1$

$\displaystyle\int_0^{\pi/4} \frac{dx}{\sin^q x \cos^p x}$ конв. ⇔ $q < 1$

⇒ $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin^p x \cos^q x}$ конв. ⇔ $p < 1$ и $q < 1$.

③ $\displaystyle\int_0^{\pi/2} \frac{\operatorname{arctg} ax}{x^\alpha}\, dx$ конв. ? $\quad \alpha > 0$, $a > 0$

$\dfrac{\operatorname{arctg} ax}{x^\alpha} \sim \dfrac{ax}{x^\alpha}$, $\quad x \to 0^+$, $\quad \displaystyle\int_0^{\pi/2} \frac{a}{x^{\alpha-1}}\, dx$ конв. ⇔ $\alpha - 1 < 1$
$\alpha < 2$

$\operatorname{arctg} ax \sim ax$

II пор. крит. $\Longrightarrow \int_0^{\pi/2} \frac{\operatorname{arctg} ax}{x^\alpha}\,dx$ конв акко $\alpha < 2$

$$\int_0^{+\infty} \frac{\operatorname{arctg} ax}{x^\alpha}\,dx = \underbrace{\int_0^{\pi/2} \frac{\operatorname{arctg} ax}{x^\alpha}\,dx}_{\text{конв. акко } \alpha<2} + \underbrace{\int_{\pi/2}^{+\infty} \frac{\operatorname{arctg} ax}{x^\alpha}\,dx}$$

$\int_{\pi/2}^{+\infty} \frac{\operatorname{arctg} ax}{x^\alpha}\,dx$: $\frac{\operatorname{arctg} ax}{x^\alpha} \sim \frac{\pi/2}{x^\alpha}$, $x \to +\infty$

$\int_{\pi/2}^{+\infty} \frac{dx}{x^\alpha}$ конв. акко $\alpha > 1$.

II пор. крит. $\Longrightarrow \int_{\pi/2}^{+\infty} \frac{\operatorname{arctg} ax}{x^\alpha}\,dx$ конв. акко $\alpha > 1$

$\Longrightarrow \int_0^{+\infty} \frac{\operatorname{arctg} ax}{x^\alpha}\,dx$ конв. акко $1 < \alpha < 2$.

$\int_a^{b} f(x)g(x)\,dx$

a — није сингуларитет

Абелов критеријум:

А1. $f \in C[a,b)$, $\int_a^b f(x)\,dx$ конв.

А2. $g \in C^1[a,b)$, монотона и ограничена на $[a,b)$

$\Rightarrow \int_a^b f(x)g(x)\,dx$ конв.

Дирихлеов критеријум:

Д1. $f \in C[a,b)$, $F(x) = \int_a^x f(t)\,dt$ огран на $[a,b)$.

Д2. $g \in C^1[a,b)$, монотона и $\lim_{x\to b^-} g(x) = 0$.

$\Rightarrow \int_a^b f(x)g(x)\,dx$ конв.

① $\int_0^{+\infty} \frac{\sin x}{x}\,dx$ конв? [проверити апс. конв.?]

у 0 фја може да се додефинише нпр са 1.

Д2. $g(x) = \frac{1}{x}$ → монотона, $C^1(0,+\infty)$, $\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to+\infty}\frac{1}{x} = 0$

$f(x) = \sin x$

Д1. $f \in C[0,+\infty)$

$$|F(x)| = \left|\int_0^x \sin t\,dt\right| = |-\cos x + \cos 0| = |1-\cos x| \le 1 + |\cos x| \le 2$$

F јесте ограничена на $[0,+\infty)$

$$\xRightarrow[\text{критеријум}]{\text{Дирихлеов}} \int_0^{+\infty} \frac{\sin x}{x}\,dx \text{ конв.}$$

Апсолутна конв?

$$\int_0^{+\infty} \frac{|\sin x|}{x}\,dx$$

$$\frac{1}{x} \ge \frac{|\sin x|}{x} \ge \frac{\sin^2 x}{x} = \frac{1-\cos 2x}{2x} = \underbrace{\frac{1}{2x}} - \underbrace{\frac{\cos 2x}{2x}}$$

$$\int_1^{+\infty} \frac{|\sin x|}{x}\,dx \ge \int_1^{+\infty}\left(\frac{1}{2x} - \frac{\cos 2x}{2x}\right)dx \overset{(*)}{=}$$ → уколико десна страна постоји у $\overline{\mathbb{R}}$.

$$\underbrace{\int_1^{+\infty} \frac{dx}{2x}}_{\substack{\text{дивергира} \\ (\to+\infty)}} - \underbrace{\int_1^{+\infty} \frac{\cos 2x}{2x}\,dx}_{\substack{\text{смена } t=2x \\ \text{конв.}}}$$

$$\frac{1}{2}\int_2^{+\infty} \frac{\cos t}{t}\,dt \leadsto f(t) = \cos t \qquad g(t) = \frac{1}{t}$$

Д2. $g \in C^1[2,+\infty)$, монотона, $\lim\limits_{t\to+\infty}\frac{1}{t} = 0$.

Д1. $f\in C[2,+\infty)$, $F(t)=\int_2^t \cos u\,du = \sin t - \sin 2$

$$|F(t)|\le 2$$

Дирихле ⇒ $\int_2^{+\infty}\frac{\cos t}{t}\,dt$ конв. ⇒ $\int_1^{+\infty}\frac{\cos 2x}{2x}\,dx$ конв.

⇒ $\int_1^{+\infty}\frac{|\sin x|}{x}\,dx$ дивергира ⇒ $\int_0^{+\infty}\frac{|\sin x|}{x}\,dx$ див.

⇒ $\boxed{\int_0^{+\infty}\frac{\sin x}{x}\,dx \text{ конв. условно, и не конв. апсолутно!}}$

(2) $\int_0^{+\infty} e^{-\frac{1}{x}}\cos x\cdot\frac{1}{\sqrt{x}}\,dx$ конв.? $\underbrace{\text{Да ли конв. апсолутно?}}_{\text{за домаћи}}$

0 ? → отклоњив синг.

$$\lim_{x\to 0^+}\underbrace{e^{-1/x}}_{\to 0}\underbrace{\cos x}_{\to 1}\frac{1}{\sqrt{x}} = \lim_{x\to 0^+}\frac{e^{-1/x}}{\sqrt{x}} \underset{t=\frac{1}{x}}{=} \lim_{t\to+\infty} e^{-t}\cdot t^{1/2} = \lim_{t\to+\infty}\frac{t^{1/2}}{e^{t}} = 0$$

$\int_0^{+\infty} e^{-1/x}\cos x\,\frac{1}{\sqrt{x}}\,dx$ конв. ⇔ $\int_1^{+\infty}\underbrace{e^{-1/x}}_{g(x)}\underbrace{\cos x\,\frac{dx}{\sqrt{x}}}_{f(x)}$ конв.

Користимо Абелов крит.:

А1. $f(x)=\frac{\cos x}{\sqrt{x}}\in C[1,+\infty)$

$\underline{\int_1^{+\infty}\frac{\cos x}{\sqrt{x}}\,dx}$ конв на основу Дирихлеовог крит.:

Д1. ✓ $\cos x\in C[1,+\infty)$, $\left|\int_1^x \cos t\,dt\right| \le 2$ → већ смо ово показали

$\qquad\qquad \left(= \sin x - \sin 1\right)$

Д2. ✓ $\frac{1}{\sqrt{x}}\in C^1[1,+\infty)$, ↘, $\lim_{x\to+\infty}\frac{1}{\sqrt{x}} = 0$

A2. $g(x) = e^{-1/x} \in C^1[1, +\infty)$

$g \uparrow$, $g > 0$

$\lim_{x \to +\infty} e^{-1/x} = 1 \Rightarrow$ g je ограничена на $[1, +\infty)$

(и то $0 < g(x) < 1$)

Абел $\Rightarrow$ $\int_1^{+\infty} e^{-1/x} \cos x \frac{1}{\sqrt{x}} dx$ конв. $\Rightarrow \int_0^{+\infty} e^{-1/x} \cos x \frac{1}{\sqrt{x}} dx$ конв.