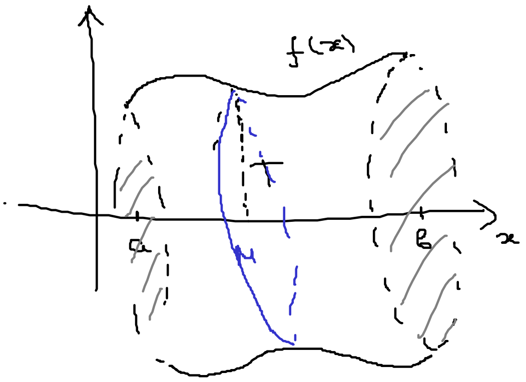
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=r^2$$

$$\left.\begin{aligned}\frac{x}{a}&=\cos t\cdot r \quad\to\quad x=ar\cos t\\ \frac{y}{b}&=\sin t\cdot r \qquad\quad y=br\sin t\end{aligned}\right\}$$

$$\int_0^{2\pi}\sqrt{a^2r^2\sin^2 t+b^2r^2\cos^2 t}\,dt = \left[r\cos t\right.$$

$$4\int_0^{\pi/2}\sqrt{\ldots} = \left[\operatorname{tg} t\right] = 4\int_0^{\infty}\sqrt{\ldots}\,\sqrt{\ldots}\,du$$

* Запремина и површина омотача обртних тела

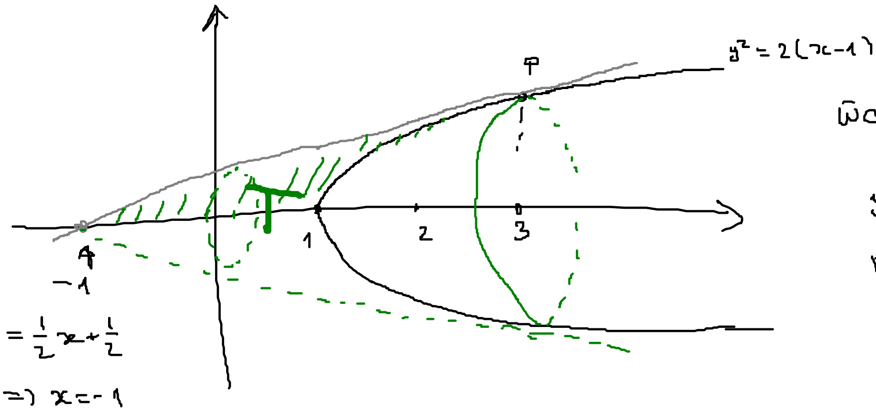


$$\mathrm{Vol}(T)=\pi\int_a^b f^2(x)\,dx$$

$$\mathrm{area}(M)=2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$

① T је добијено ротацијом површи ограниченом са $y^2=2(x-1)$ и тангентом на ову параболу у $P(3,2)$ око x-осе.

$$\mathrm{Vol}(T)=\mathrm{Vol}(\overset{\text{купа}}{K})-\mathrm{Vol}(\overset{\text{параболоид}}{U}$$



$0=\frac{1}{2}x+\frac{1}{2}$

$\Rightarrow x=-1$

тангента у P:

$y=kx+n$

$k=f'(3)$

x- коорд. тачке P

$f=?$ $\quad f(x)=\sqrt{2(x-1)}$

$$f'(x)=\frac{1}{2\sqrt{2(x-1)}}\cdot 2$$

$$f'(3)=\frac{1}{\sqrt{2\cdot 2}}=\frac{1}{2}$$

решење: $2=\underset{k}{\frac{1}{2}}\cdot 3+n \longrightarrow n=\frac{1}{2}$

$t:\quad y=\frac{1}{2}x+\frac{1}{2}$

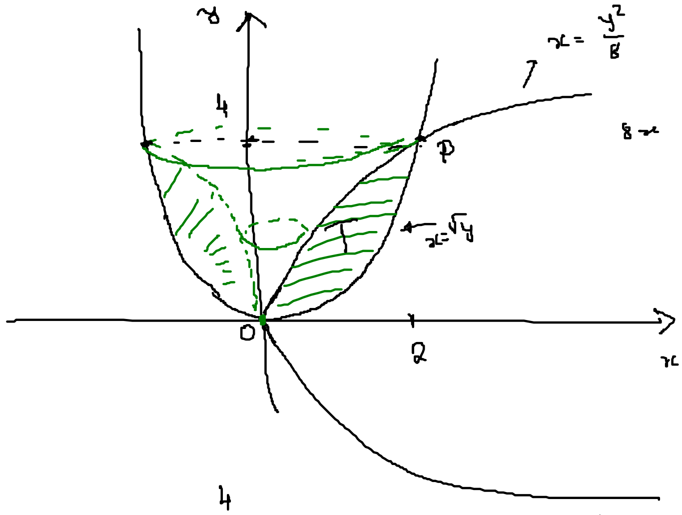
$$Vol(K) = \pi \int_{-1}^{3} \left(\frac{1}{2}x+\frac{1}{2}\right)^2 dx = \pi\, 2 \cdot \frac{\left(\frac{1}{2}x+\frac{1}{2}\right)^3}{3}\Bigg|_{-1}^{3} = 2\pi\, \frac{2^3}{3} = \frac{16\pi}{3}$$

$$Vol(U) = \pi \int_{1}^{3} \left(\sqrt{2(x-1)}\right)^2 dx = \pi \int_1^3 2(x-1)\, dx = 2\pi\, \frac{x^2}{2}\Big|_1^3 - 2\pi\cdot(3-1) = 8\pi - 4\pi = 4\pi$$

$$Vol(T) = \frac{16\pi}{3} - 4\pi = \frac{4\pi}{3}$$

② Одредити запремину тела T добијеног ротацијом око y-осе површи $y^2 = 8x$, $x^2 = y$.

$$Vol(T) = Vol(U) - Vol(K)$$



$P\ (x, y)$

$x = \frac{y^2}{8}$

$x = \sqrt{y}$

$\frac{y^2}{8} = \sqrt{y} \ \uparrow^2 \qquad \frac{y^3}{8^2} = 1$

$y^3 = 4^3$

$y = 4$

$y = 0$

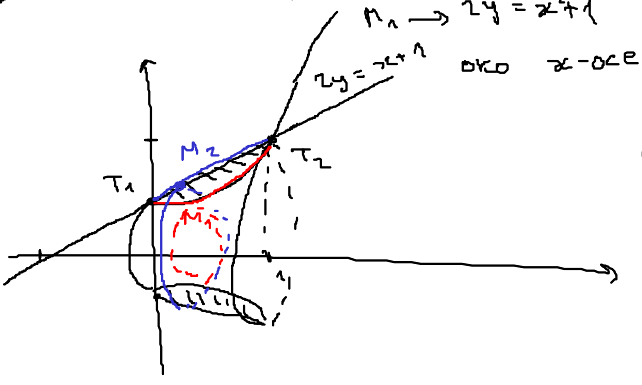
$$Vol(U) = \pi \int_0^4 \underbrace{(\sqrt{y})^2\, dy}_{\text{зато што ротирамо око } y\text{-осе}} = \pi \int_0^4 y\, dy = \pi\, \frac{16}{2} = 8\pi$$

$$Vol(K) = \pi \int_0^4 \frac{y^4}{\underset{8^2}{64}}\, dy = \pi \cdot \frac{y^5}{64\cdot 5}\Big|_0^4 = \pi\, \frac{4^5}{64\cdot 5} = \pi\, \frac{16}{5} \qquad 4^5 = 64\cdot 16$$

$$Vol(T) = 8\pi - \frac{16}{5}\pi$$

③ area T = ?, $T: M_2 \to 2y = x+1$, $M_1 \to 2y = x^2+1$ око x-осе ротира

T_1	T_2
$x = 0$	$x = 1$
$y = \frac{1}{2}$	$y = 1$



area T = area M_1 + area M_2

$$\text{area } M_1 = 2\pi \int_0^1 \left(\frac{x^2+1}{2}\right)\sqrt{1 + x^2}\, dx = \pi \int_0^1 (x^2+1)^{3/2} dx = \left[\, x = tg\, t,\ dx = \frac{dt}{\cos^2 t},\ t: 0 \to \frac{\pi}{4} \right]$$

$$= \pi \int_0^{\pi/4} (tg^2 t + 1)^{3/2} \frac{dt}{\cos^2 t} = \pi \int_0^{\pi/4} \frac{1}{(\cos^2 t)^{3/2}} \frac{dt}{\cos^2 t} = \pi \int_0^{\pi/4} \frac{dt}{\cos^5 t} \cdot \frac{\cos t}{\cos t}$$

$\downarrow$

$u = \sin t$

$\dots$

$$\text{area } M_2 = 2\pi \int_0^1 \left(\frac{x+1}{2}\right) \sqrt{1 + \frac{1}{4}} \, dx = \dots$$

Несвојствени интеграли

$\int_a^{+\infty}$, $\int_{-\infty}^{b}$, $\int_{-\infty}^{+\infty}$, $\int_a^b f(x)$, $a, b \in \overline{\mathbb{R}}$, f није деф у b ($b \in \overline{\mathbb{R}}$)

f деф у a

f није дефинисано у некој од граница

$a < b$

$$\int_a^b f(x)dx := \lim_{\beta \to b^-} \int_a^\beta f(x)dx$$

$\longrightarrow$ уколико овај лимес постоји у $\underline{\underline{\mathbb{R}}}$ (без ∞) онда $\int_a^b f(x)dx$ КОНВЕРГИРА

$\longrightarrow$ уколико овај лимес не постоји у $\mathbb{R}$ онда $\int_a^b f(x)dx$ ДИВЕРГИРА

1) $p \in \mathbb{R}$

$$\int_0^1 \frac{dx}{x^p} = ?$$

$\frac{1}{x^p} \to$ није увек дефинисано у 0

$p \leq 0 \to \frac{1}{x^p}$ могу да додефинишем непр. у 0

$$\int_0^1 \frac{dx}{x^p} = \int_0^1 x^{-p} dx = \frac{x^{1-p}}{1-p}\Big|_0^1 = \frac{1}{1-p}$$

$p > 0 \to$ имамо несвојствен интеграл

$p \neq 1$

$$\lim_{\beta \to 0^+} \int_\beta^1 \frac{dx}{x^p} = \lim_{\beta \to 0^+} \left(\frac{x^{1-p}}{1-p} \Big|_\beta^1 \right) = \lim_{\beta \to 0^+} \frac{1}{1-p} - \frac{\beta^{1-p}}{1-p} = \frac{1}{1-p} - \lim_{\beta \to 0^+} \frac{\beta^{1-p}}{1-p}$$

$p \neq 1$

константа

$\int_0^1 \frac{dx}{x^p}$

ово је коначно за $p < 1$

за $p > 1$ дивергира

$\Rightarrow \int_0^1 \frac{dx}{x^p}$ конвергира за $p < 1$

дивергира за $p > 1$

$$p = 1: \lim_{\beta \to 0^+} \int_\beta^1 \frac{dx}{x} = \lim_{\beta \to 0^+} \ln x \Big|_\beta^1 = \lim_{\beta \to 0^+} -\ln\beta = +\infty \to \text{дивергира}$$

$$\int_0^1 \frac{dx}{x^p} \quad \text{конвергира акко } p<1$$
$$\text{дивергира акко } p \geq 1$$

ⓑ → сингуларитет (f није деф у b)

$$\int_a f(x)dx \underset{\downarrow}{=} \lim_{\beta\to b^-} F(x) - F(a) \longrightarrow \text{„УОПШТЕЊЕ НЈУТН-ЛАЈБНИЦА"}$$

$$F(x)=\int f(x)dx$$

②
$$\int_1^{+\infty} \frac{dx}{x^p} = ?$$
$p\in\mathbb{R}$

$p\neq 1$
$$\int_1^{+\infty} \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}\Big|_1^{\infty} = \lim_{x\to+\infty} \frac{x^{1-p}}{1-p} - \frac{1}{1-p} = \begin{cases} 0-\frac{1}{1-p}, & p>1 \\ +\infty & p<1 \end{cases}$$

$$\int_1^{\infty} \frac{dx}{x^p} \quad \text{конвергира ако } p>1$$
$$\text{дивергира ако } p<1$$

$p=1$
$$\int_1^{\infty} \frac{dx}{x} = \ln x\Big|_1^{\infty} = \lim_{x\to+\infty} \ln x - \ln 1 = +\infty \to \int_1^{+\infty} \frac{dx}{x} \text{ дивергира}$$

$$\int_1^{+\infty} \frac{dx}{x^p} \text{ конв акко } p>1.$$

③
$$\int_{ⓑ}^1 \ln x = \lim_{\beta\to 0^+} \int_\beta \ln x\,dx = \left[\begin{matrix} u=\ln x & \to du=\frac{dx}{x} \\ dv=dx & \to v=x \end{matrix}\right] = \lim_{\beta\to 0^+}\left(x\ln x\Big|_\beta^1 - \int_\beta^1 dx\right)$$

$$= \lim_{\beta\to 0^+} \underbrace{-\beta\cdot\ln\beta}_{\to 0} - (1-\beta) = \underline{\underline{-1}}$$

$$\lim_{\beta\to 0^+} \frac{\ln\beta}{\frac{1}{\beta}} \overset{*}{\underset{\text{ЛП}}{=}} \lim_{\beta\to 0^+} \frac{\frac{1}{\beta}}{-\frac{1}{\beta^2}} = \lim_{\beta\to 0^+} -\beta = 0$$

$$\Rightarrow \int_0^1 \ln x\,dx \to \text{конв.}$$

4. $$\int_1^2 \frac{1}{x\ln x}\,dx = \left[\begin{array}{l} t=\ln x \\ dt=\frac{dx}{x}\end{array}\right] = \int_0^{\ln 2}\frac{dt}{t} = \underbrace{\int_0^1 \frac{dt}{t}}_{\text{дивергира}} + \underbrace{\int_1^{\ln 2}\frac{dt}{t}}_{\text{конв.}} \to \text{дивергира.}$$

$$\underbrace{\int_a^b f(x)\,dx}_{\text{конв.}} + \underbrace{\int_c^d g(x)\,dx}_{\text{конв.}} \to \text{конв.}$$

$$\underbrace{\int_a^b f(x)\,dx}_{\text{див.}} + \underbrace{\int_c^d g(x)\,dx}_{\text{конв.}} \to \text{див.}$$

$$\underbrace{\int_a^b f(x)\,dx}_{\text{див.}} + \underbrace{\int_c^d g(x)\,dx}_{\text{див.}} \to \text{не знамо шта се дешава!}$$

(као код лимеса проблем је у случају $\underbrace{\infty - \infty}_{\text{бесконачности истог знака}}$)

5. $$\int_0^1 \frac{x-\operatorname{arctg} x}{x^2(1+x^2)}\,dx = \int_0^1 \frac{x}{x^2(1+x^2)}\,dx - \int_0^1 \frac{\operatorname{arctg} x}{x^2(1+x^2)}\,dx \to$$

за вежбу показати да су оба интеграла $+\infty$.

$\int_0^1 \frac{x}{x^2(1+x^2)}\,dx$ → див. јер се подинт. ф-ја понаша као $1/x$

$\int_0^1 \frac{\operatorname{arctg} x}{x^2(1+x^2)}\,dx$ → $\frac{1}{x}$ → испостави се да и ово дивергира

дакле не можемо овако

$$\int_0^1 \frac{x-\operatorname{arctg} x}{x^2(1+x^2)}\,dx = \lim_{\beta\to 0^+}\int_\beta^1 \frac{x-\operatorname{arctg} x}{x^2(1+x^2)}\,dx \overset{(*)}{=}$$

$$\int_\beta^1 \frac{x-\operatorname{arctg} x}{x^2(1+x^2)}\,dx = \int_\beta^1 \frac{x\,dx}{x^2(1+x^2)} - \int_\beta^1 \frac{\operatorname{arctg} x}{x^2(1+x^2)}\,dx$$

$$\int_\beta^1 \frac{dx}{x(1+x^2)} \underset{*}{=} \int_\beta^1 \frac{dx}{x} - \int_\beta^1 \frac{x\,dx}{1+x^2} = \ln x\Big|_\beta^1 - \frac{1}{2}\ln(1+x^2)\Big|_\beta^1 = -\ln\beta - \frac{1}{2}\ln 2 + \frac{1}{2}\ln(\beta^2+1)$$

* ┌

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{Ax^2+A+Bx^2+Cx}{x(1+x^2)}$$

$A+B=0$

$C=0$

$A=1$

$B=-1$ ┘

$$\int_\beta^1 \frac{\operatorname{arctg} x\,dx}{x^2(1+x^2)} = \left\ulcorner \begin{array}{l} t=\operatorname{arctg} x \\ dt=\frac{dx}{1+x^2} \end{array}\right\lrcorner = \int_{\operatorname{arctg}\beta}^{\pi/4} \frac{t\,dt}{tg^2 t} = \left\ulcorner \begin{array}{l} u=t \to du=dt \\ dv=\frac{dt}{tg^2 t} \to v=-ctg\,t - t \end{array}\right\lrcorner$$

┌

$$\int \frac{dt}{tg^2 t} = \int \frac{1-\sin^2 t}{\sin^2 t}\,dt = \int \frac{dt}{\sin^2 t} - \int dt = -ctg\,t - t + C$$ ┘

$$= t\cdot(-ctg\,t - t)\Big|_{\operatorname{arctg}\beta}^{\pi/4} + \int_{\operatorname{arctg}\beta}^{\pi/4} (\overset{\frac{\cos t}{\sin t}}{ctg\,t} + t)\,dt$$

$$= \frac{\pi}{4}\left(-1-\frac{\pi}{4}\right) - \operatorname{arctg}\beta\left(-\frac{1}{\beta} - \operatorname{arctg}\beta\right) + \int_{\operatorname{arctg}\beta}^{\pi/4} \frac{\cos t}{\sin t}\,dt + \int_{\operatorname{arctg}\beta}^{\pi/4} t\,dt$$

$ctg = \frac{1}{tg}$ $\qquad u = \sin t$

$$= \frac{\pi}{4}\left(-1-\frac{\pi}{4}\right) + \frac{\operatorname{arctg}\beta}{\beta} + \operatorname{arctg}^2\beta + \ln|\sin t|\Big|_{\operatorname{arctg}\beta}^{\pi/4} + \frac{t^2}{2}\Big|_{\operatorname{arctg}\beta}^{\pi/4}$$

$$= \frac{\pi}{4}\left(-1-\frac{\pi}{4}\right) + \frac{\operatorname{arctg}\beta}{\beta} + \frac{1}{2}\operatorname{arctg}^2\beta + \ln\frac{\sqrt{2}}{2} - \ln(\operatorname{arctg}\beta) + \frac{\pi^2}{32}$$

$$\int_0^1 \frac{x-\operatorname{arctg} x}{x^2(1+x^2)}\,dx = \lim_{\beta\to 0^+}\Big(-\ln\beta - \cancel{\tfrac{1}{2}\ln 2} + \underbrace{\tfrac{1}{2}\ln(\beta^2+1)}_{\to 0} - \frac{\pi}{4}\left(-1-\frac{\pi}{4}\right) - \underbrace{\frac{\operatorname{arctg}\beta}{\beta}}_{\to 1}$$

$$- \underbrace{\tfrac{1}{2}\operatorname{arctg}^2\beta}_{\to 0} + \cancel{\tfrac{1}{2}\ln 2} + \ln(\operatorname{arctg}\beta) - \frac{\pi^2}{32}\Big)$$

$$= +\frac{\pi}{4} + \frac{\pi^2}{16} - 1 - \frac{\pi^2}{32} + \lim_{\beta\to 0^+}\left(\ln(\operatorname{arctg}\beta) - \ln\beta\right)$$

$$= \frac{\pi}{4} + \frac{\pi^2}{32} - 1 + \underbrace{\lim_{\beta\to 0^+} \ln \underbrace{\frac{\operatorname{arctg}\beta}{\beta}}_{\to 1}}_{=0} = \frac{\pi}{4} + \frac{\pi^2}{32} - 1$$