$*\ \sigma(f,P,\xi) = \sum_{k=1}^{n} (x_k - x_{k-1}) f(\xi_k)$, $f:[a,b]\to\mathbb{R}$

$P = \{ a \overset{(\le)}{=} x_0 < x_1 < x_2 < \dots < x_n \overset{(\le)}{=} b \}$

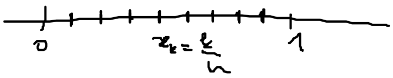
$\xi = \{ \xi_k \in [x_{k-1}, x_k] \}$

$\lambda(P) = \max\limits_{1\le k\le n} (x_k - x_{k-1})$

$\int_a^b f(x)dx \overset{(*)}{=} \lim\limits_{\lambda(P)\to 0} \sigma(f,P,\xi)$, (*) важи уколико lim постоји у $\mathbb{R}$.

(*) нпр. ако је $f\in\mathcal{R}[0,1]$ (дакле, lim у (*) постоји у $\mathbb{R}$)

P_n: $x_k = \frac{k}{n}$



ξ^n: $\xi_k = \frac{k}{n}$

$\lambda(P_n) = \max\limits_{1\le k\le n} (\overset{=\frac{k}{n}-\frac{k-1}{n}}{x_k - x_{k-1}}) = \max\limits_{1\le k\le n} \frac{1}{n} = \frac{1}{n}$

$\Rightarrow \lim\limits_{n\to\infty} \sigma(f,P_n,\xi^n) = \lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx$.

---

① $\chi(x) = \begin{cases} 0, & x\notin\mathbb{Q} \\ 1, & x\in\mathbb{Q}\end{cases}$ неће бити Риман интеграбилна на $[0,1]$

P произвољна подела интервала $[0,1]$

P : $0 = x_0 < x_1 < x_2 < \dots < x_n = b$

ξ : $\xi_k \in (x_{k-1}, x_k)\setminus\mathbb{Q}$ → (јер је скуп ирац. бр. густ у $\mathbb{R}$)

η : $\eta_k \in (x_{k-1}, x_k)\cap\mathbb{Q}$ → (јер је $\mathbb{Q}$ густ у $\mathbb{R}$)

$\sigma(\chi, P, \xi) = \sum_{k=1}^{n} (x_k - x_{k-1}) \overset{=0}{\chi(\xi_k)} = 0$

$\sigma(\chi, P, \eta) = \sum_{k=1}^{n} (x_k - x_{k-1}) \underset{1}{\underbrace{\chi(\eta_k)}} = \sum_{k=1}^{n} (x_k - x_{k-1}) = (x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + \dots + x_n - x_{n-1}) =$
$= x_n - x_0 = 1$

$\lim\limits_{\lambda(P)\to 0} \sigma(\chi, P, A)$ → не постоји јер зависи од скупа A

$\Rightarrow \chi$ није Риман интеграбилна на $[0,1]$!

② $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2\left(\left(\frac{k}{n}\right)^2+1\right)} = \lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\left(\frac{k}{n}\right)^2+1} \overset{\star}{=} \int_0^1 \frac{1}{x^2+1}dx = \operatorname{arctg}x\Big|_0^1 = \frac{\pi}{4}$

$f(x) = \frac{1}{x^2+1}$
$f\in\mathcal{R}[0,1]$

③ $\lim\limits_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 \ln\left(1+\frac{k}{n}\right) = \lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \ln\left(1+\frac{k}{n}\right) = \int_0^1 x^2\ln(1+x)dx$

$f(x) = x^2\ln(1+x)$
$f\in\mathcal{R}[0,1]$ $= \underbrace{\dots}_{\text{за вежбу}}$

④ $\lim\limits_{n\to\infty} \frac{1}{2n^2}\sum\limits_{k=0}^{n-1}(2k+1)\operatorname{arctg}\frac{2k+1}{2n} = \lim\limits_{n\to\infty}\underbrace{\frac{1}{n}}_{x_{k+1}-x_k}\sum\limits_{k=0}^{n-1}\underbrace{\frac{2k+1}{2n}}_{\xi_k}\operatorname{arctg}\frac{2k+1}{2n} =$

$f(x) = x\operatorname{arctg}x$

$f\in R[0,1]$

$\xi_k : \frac{2k+1}{2n}, \quad 0\le k\le n-1$

$x_{k+1}-x_k = \frac{1}{n}, \quad x_k = \frac{k}{n} \qquad ?\ \xi_k \in (x_k, x_{k+1})?$ ✓

$\frac{2k+1}{2n} \in \left(\frac{k}{n}, \frac{k+1}{n}\right)?$, $\frac{k}{n} = \frac{2k}{2n}$, $\frac{k+1}{n} = \frac{2k+2}{2n}$

$P_n: \ x_k = \frac{k}{n} \qquad 0\le k\le n$

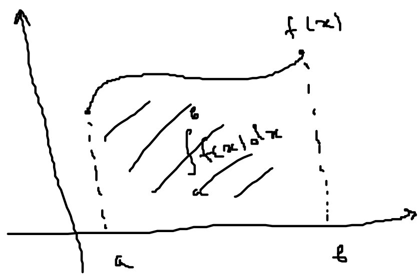
$\xi : \ \xi_k = \frac{2k+1}{2n} \qquad 0\le k\le n-1$

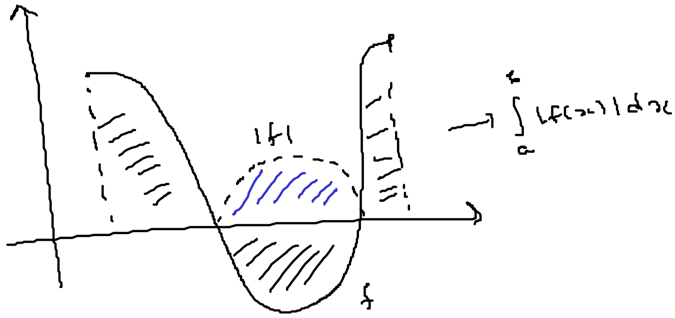
$f(x) = x\operatorname{arctg}x \in R[0,1]$

$\lambda(P_n) = \frac{1}{n} \to 0$

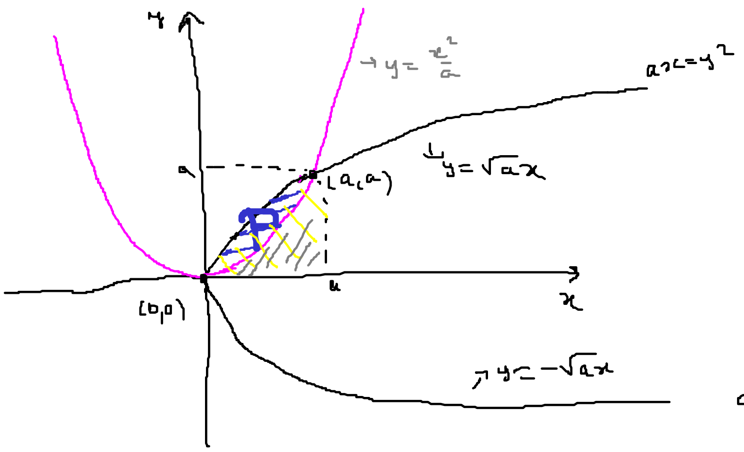
$\lim\limits_{n\to\infty}\sigma(f,P_n,\xi) = \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=0}^{n-1}\frac{2k+1}{2n}\operatorname{arctg}\frac{2k+1}{2n} = \int_0^1 x\cdot\operatorname{arctg}x\,dx =$ за вежбу

* Површина равних области





① Наћи површину области P која је ограничена кривама $ax=y^2$, $ay=x^2$, $a>0$



area P = ?

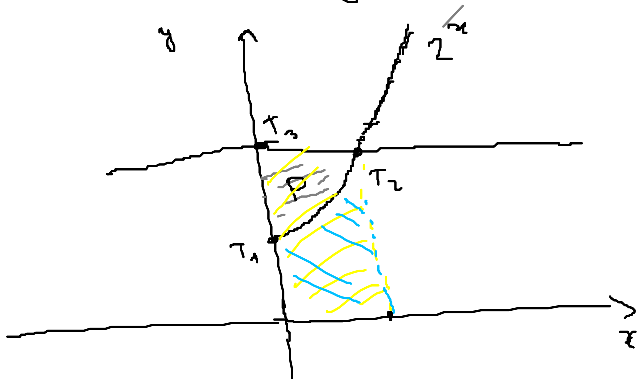
$ax=y^2 \to x=\frac{y^2}{a}$

$ay=x^2 \to ay=\frac{y^4}{a^2} \to a^3y=y^4 \to y=0,\ x=0; \quad \to y=a,\ x=a$

$\text{area}P = \int_0^a \sqrt{ax}\,dx - \int_0^a \frac{x^2}{a}dx$

$= \sqrt{a}\cdot\frac{x^{3/2}}{3/2}\Big|_0^a - \frac{x^3}{3a}\Big|_0^a = \frac{a^2}{3}$

(2) Одредити површину области P ограничене са $y = 2^x$, $y=2$, $x=0$



$T_1(0,1) \sim y=2^x,\ x=0$ ($1 = 2^0$)

T_2: $y=2^x$, $y=2$ $\to T_2(1,2)$

T_3: $y=2$, $x=0$ $\quad T_3(0,2)$

$$\text{area}P = \int_0^1 2-2^x\,dx = 2\int_0^1 dx - \int_0^1 \underbrace{2^x}_{=e^{x\ln 2}}\,dx$$

$$= 2 - \underbrace{\frac{e^{x\cdot\ln 2}}{\ln 2}\Big|_0^1}_{} = 2 - \left(\frac{2}{\ln 2} - \frac{1}{\ln 2}\right)$$

$$= 2 - \frac{1}{\ln 2}$$

(3) —||— P: $x^2+y^2=8$, $x^2=-2y$, $\underline{y\le 0}$

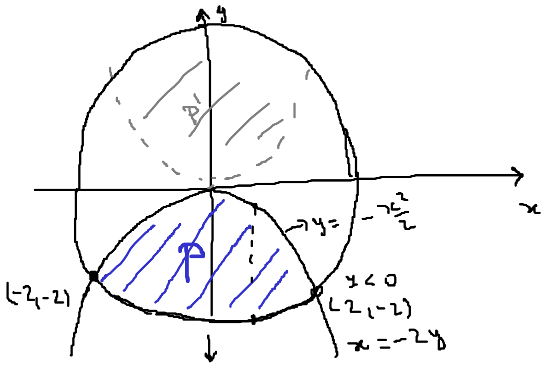
$$\begin{cases} x^2+y^2=8 \\ x^2=-2y \Rightarrow y\le 0 \end{cases}$$

$y^2-2y=8$

$y^2-2y+1=9$

$(y-1)^2=3^2 \to$ $y-1=3 \to y=4$

$y-1=-3 \to \boxed{y=-2}$



$y = -\sqrt{8-x^2}$

$x^2=-2y=4 \to x=-2$, $x=2$

$f_1, f_2 < 0 \quad f_1 < f_2$

$-f_1 > -f_2$

$-f_1 - (-f_2) = f_2 - f_1$

$$\text{area}P = \int_{-2}^{2} -\frac{x^2}{2} - \left(-\sqrt{8-x^2}\right)dx = \underbrace{\int_{-2}^{2} -\frac{x^2}{2}dx}_{=-\frac{8}{3}} + \int_{-2}^{2}\sqrt{8-x^2}\,dx$$

$x = 2\sqrt2 \sin t$

$t\in ?$

$t = \arcsin\frac{x}{2\sqrt2}$

$t(-2) = \arcsin -\frac{2}{2\sqrt2} = -\frac{\pi}{4}$

$t(2) = \arcsin\frac{2}{2\sqrt2} = \frac{\pi}{4}$

$$= -\frac{8}{3} + \int_{-\pi/4}^{\pi/4} 2\sqrt2\cos t \cdot 2\sqrt2 \cos t\,dt = -\frac{8}{3} + 16\int_0^{\pi/4} \underbrace{\cos^2 t}_{=\frac{1+\cos 2t}{2}}\,dt$$

$$= -\frac{8}{3} + 16\cdot\frac{\pi}{8} + \frac{16}{4}\cdot \sin 2t\Big|_0^{\pi/4}$$

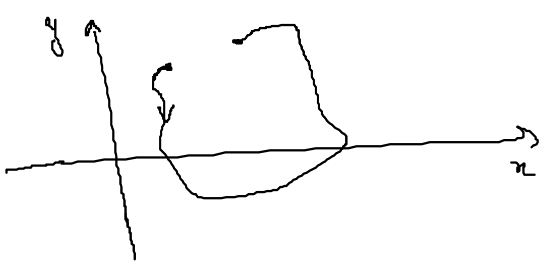
$$= -\frac{8}{3} + 2\pi + 4 = 2\pi + \frac{4}{3}$$

* дужина лукова

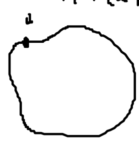
$x = \varphi(t)$

$y = \psi(t)$

$t\in[\alpha,\beta]$



$(\varphi(\alpha),\psi(\alpha)) = (\varphi(\beta),\psi(\beta))$



$$l = \int_{\alpha}^{\beta} \sqrt{\varphi'(t)^2 + \psi'(t)^2}\, dt$$

$\ulcorner \varphi(t) = t$, $x = t$

$y = f(x)$

$x \in [a, b]$

$$l = \int_{a}^{b} \sqrt{1 + f'(x)^2}\, dx \lrcorner$$

① Израчунати дужину лука $y = \underbrace{\ln(\cos x)}_{f(x)}$, $0 \le x \le \frac{\pi}{4}$

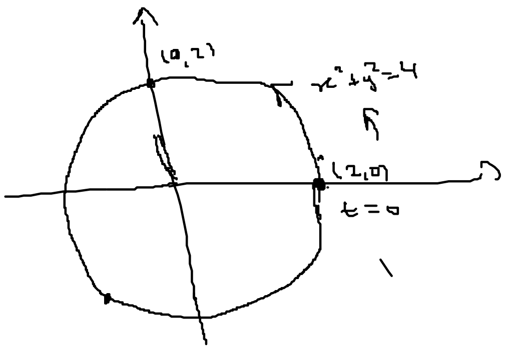
$$l = \int_0^{\pi/4} \sqrt{1 + f'(x)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + tg^2 x}\, dx = \int_0^{\pi/4} \frac{dx}{\cos x} \cdot \frac{\cos x}{\cos x} = \dots$$

$$f'(x) = \frac{1}{\cos x} \cdot \sin x$$

$$= \int_0^{\pi/4} \frac{\cos x\, dx}{1 - \sin^2 x} = \ulcorner u = \sin x \lrcorner = \int_0^{\frac{1}{\sqrt{2}}} \frac{du}{1 - u^2}$$

$$= \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| \Big|_0^{1/\sqrt{2}} = \frac{1}{2} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right)$$

② Одредити обим круга $x^2 + y^2 = 4 \Rightarrow r = 2$



$x = 2\cos t = \varphi(t)$

$y = 2\sin t = \psi(t)$

$x^2 + y^2 = 4$

$0 \le t \le 2\pi$

$$O = \int_0^{2\pi} \sqrt{\varphi'(t)^2 + \psi'(t)^2}\, dt = \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t}\, dt$$

$$= 2 \int_0^{2\pi} dt = 4\pi$$