* Теореме о средњој вредности (Т о ср. вр.)

(I) $f, g \in R[a,b]$, $m = \inf_{x \in [a,b]} f(x)$, $M = \sup_{x \in [a,b]} f(x)$, $g(x) \geq 0$ $\forall x \in [a,b]$

$\Rightarrow \exists \mu \in [m, M]$ $\int_a^b f(x) g(x)\,dx = \mu \int_a^b g(x)\,dx$

$f \in C[a,b] \Rightarrow \exists \xi \in [a,b]$ $\int_a^b f(x) g(x)\,dx = f(\xi) \int_a^b g(x)\,dx$

ако је још $g(x) = 1$, $\forall x \in [a,b] \Rightarrow \exists \xi \in [a,b]$ $\int_a^b f(x)\,dx = f(\xi) \cdot (b-a)$

(II) • $f \in C[a,b]$, $g \nearrow$, $g(x) \geq 0$ $\forall x \in [a,b]$, $g \in C^1[a,b]$

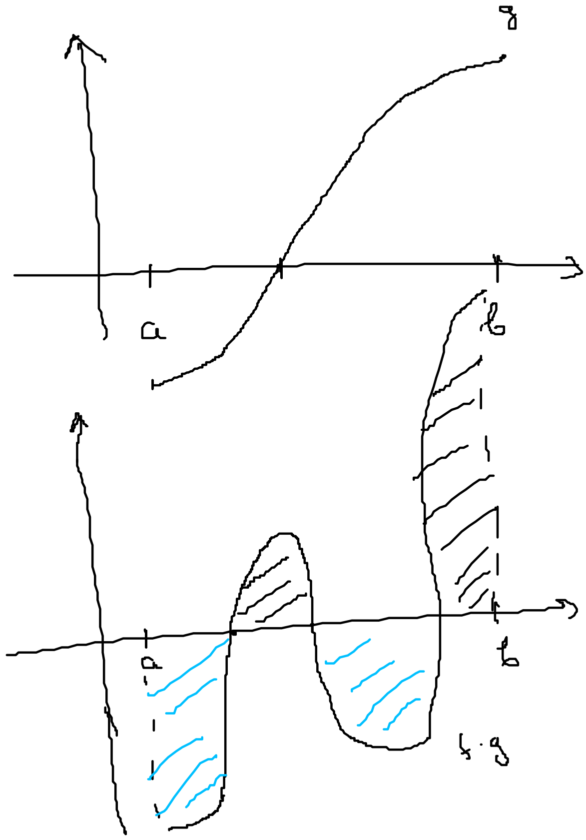
$\Rightarrow \exists \xi \in [a,b]$ $\int_a^b f(x) g(x)\,dx = g(b) \int_\xi^b f(x)\,dx$

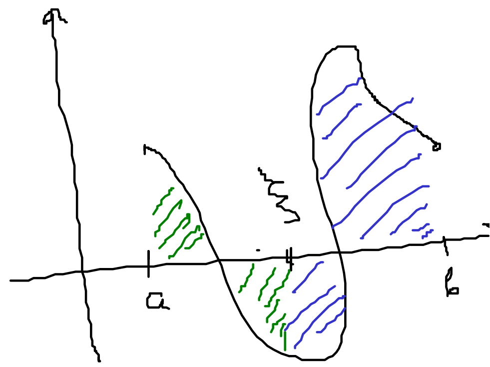
• $f \in C[a,b]$, $g \searrow$, $g(x) \geq 0$, $g \in C^1[a,b]$

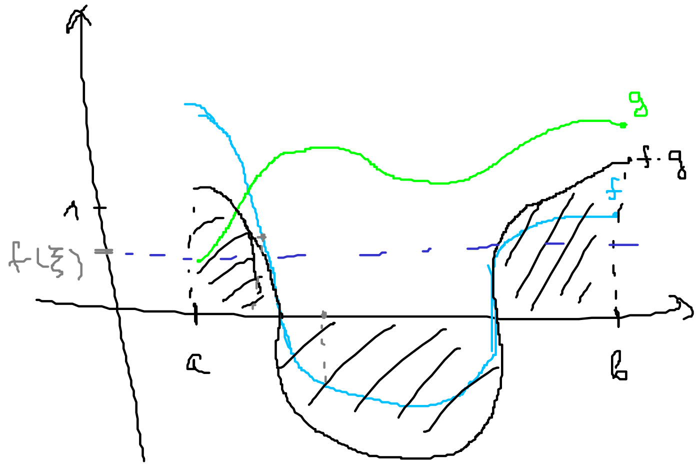
$\Rightarrow \exists \xi \in [a,b]$ $\int_a^b f(x) g(x)\,dx = g(a) \int_a^\xi f(x)\,dx$

(III) $f \in C[a,b]$, g монотона, $g \in C^1[a,b]$

$\Rightarrow \exists \xi \in [a,b]$ $\int_a^b f(x) g(x)\,dx = g(a) \int_a^\xi f(x)\,dx + g(b) \int_\xi^b f(x)\,dx$







① $f\in C(\mathbb{R})$, $\lim\limits_{x\to+\infty} f(x)=A>0$, $\lim\limits_{n\to\infty}\int_0^1 f(nx)\,dx = ?$

$$\int_0^1 f(nx)\,dx = \left[\begin{aligned} t&=nx \\ dt&=dx\cdot n\end{aligned}\right] = \frac{1}{n}\int_0^n f(t)\,dt$$

$$\lim_{n\to\infty}\frac{\int_0^n f(t)\,dt}{n} \overset{\text{ШТ}}{=} \lim_{n\to\infty}\frac{\int_0^{n+1} f(t)\,dt-\int_0^n f(t)\,dt}{n+1-n} = \lim_{n\to\infty}\int_n^{n+1} f(t)\cdot 1\,dt$$

$$= \lim_{n\to\infty} f(\xi_n)\cdot(n+1-n) = \lim_{n\to\infty} f(\xi_n) = A.$$

(I) Т о ср. вр.

$\xi_n\in[n,n+1] \Rightarrow \xi_n\to+\infty$

② $\lim\limits_{n\to\infty}\int_n^{n+p}\frac{\sin x}{x}\,dx = ?$, $p>0$

$$\int_n^{n+p}\frac{\sin x}{x}\,dx = \left[\begin{aligned} &f(x)=\sin x\in C[a,b] \\ &g(x)=\tfrac{1}{x} \\ &g\downarrow,\ g\ge 0,\ g\in C^1[a,b]\end{aligned}\right] \overset{\text{II Т о ср. вр. } \xi_n}{=} \frac{1}{n}\int_n \sin x\,dx$$

$\exists\,\xi_n\in[n,n+p]$

$$= \frac{1}{n}\cdot\left(-\cos x\right)\Big|_n^{\xi_n} = \frac{\cos n-\cos\xi_n}{n}$$

$$\left|\int_n^{n+p}\frac{\sin x}{x}\,dx\right| = \frac{|\cos n-\cos\xi_n|}{n} \underset{|\cos x|\le 1}{\le} \frac{2}{n}\xrightarrow[n\to\infty]{} 0$$

$T \text{ o } 2\Pi \Rightarrow \lim_{n\to\infty} \int_n^{n+p} \frac{\sin x}{x}\,dx = 0.$

③ $f \in C(\mathbb{R})$, $\forall x \in \mathbb{R}, n \in \mathbb{N}$: $\int_0^{x+\frac{1}{n}} f(t)\,dt = \int_0^x f(t)\,dt + \frac{f(x)}{n}$

Одредити фју f.

$$\overbrace{\underbrace{\int_0^{x+\frac{1}{n}} f(t)\,dt}_{\text{диференц. фја по } x} - \underbrace{\int_0^x f(t)\,dt}_{\text{диф. фја по } x}}^{G(x)} = \frac{f(x)}{n}$$

f непр $\Rightarrow$ $\Longrightarrow$ f је диференцијабилна

f непр $\Rightarrow$ лева страна има непрекидан извод $\Rightarrow f \in C^1 \Rightarrow f' \in C(\mathbb{R})$

$$G(x) = \int_x^{x+\frac{1}{n}} f(t)\,dt = \frac{f(x)}{n}$$

$$\int_x^{x+\frac{1}{n}} f(t)\,dt = f(\xi_n)\cdot\left(x+\frac{1}{n}-x\right) = \frac{f(x)}{n}$$

I T о ср. вр.

$\xi_n \in [x, x+\frac{1}{n}]$

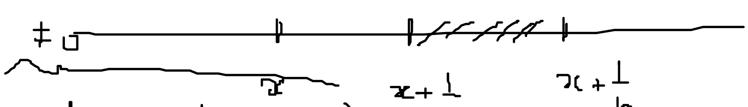
$\Rightarrow f(x) = f(\xi_n)$

x фиксирано

$\forall n \in \mathbb{N}$, $m > n$

$$\int_x^{x+\frac{1}{n}} f(t)\,dt = \frac{f(x)}{n}$$

$$-\downarrow \quad \int_x^{x+\frac{1}{m}} f(t)\,dt = \frac{f(x)}{m}$$



$$\int_{x+\frac{1}{m}}^{x+\frac{1}{n}} f(t)\,dt = f(x)\left(x+\frac{1}{n} - \left(x+\frac{1}{m}\right)\right)$$

$$\underbrace{}_{\substack{= \\ \text{I T о ср. вр.}}} f(\xi_{n,m})\cdot\left(x+\frac{1}{n} - \left(x+\frac{1}{m}\right)\right), \quad \xi_{n,m} \in \left[x+\frac{1}{m}, x+\frac{1}{n}\right]$$

$f(x) = f(\xi_{n,m})$

$m = n+1 \qquad y_n = \xi_{n,n+1} \in \left[x + \frac{1}{n+1}, x + \frac{1}{n}\right]$

$y_n \neq x \quad , \quad \lim_{n\to\infty} y_n = x$

$f(y_n) = f(x) \overset{\text{Рол}}{\Longrightarrow} \exists z_n \in (x, y_n) \quad f'(z_n) = 0$

$\lim_{n\to\infty} z_n = x \quad \Rightarrow \quad 0 = \lim_{n\to\infty} f'(z_n) = f'(x)$

$f' \in C(\mathbb{R})$

$f'(x) = 0$

$\Rightarrow \forall x \in \mathbb{R} \ f'(x) = 0 \quad \Rightarrow \quad \forall x \in \mathbb{R} \quad f(x) = \text{const.}$

$f: [a,b] \to \mathbb{R}$

$* \quad \sigma(f, P, \xi) = \sum_{k=0}^{n-1} (\underbrace{x_{k+1} - x_k}_{=\frac{1}{n}}) \, f(\xi_k), \quad P = \{ x_k : a = x_0 < x_1 < \dots < x_n = b\}$

$f(x_k) \qquad \xi_k = \frac{2k+1}{2n}$

$\xi = \{ \xi_k : \xi_k \in [x_k, x_{k+1}]\}$

$\int_a^b f(x)\,dx = \lim_{\lambda(P)\to 0} \sigma(f, P, \xi)$

$\lambda(P) = \max_k (x_k - x_{k-1}) = \frac{1}{n}$

f Риман интеград. ако $\lim_{\lambda(P)\to 0} \sigma(f, P, \xi)$ постоји у $\mathbb{R}$.

1. $\chi(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$ $\to$ има непребројиво много прекида у свакој тачки у $\mathbb{R}$

χ није Риман интеградилна на $[0,1]$!

Δ: Нека је $P = \{ x_k : x_0 = 0 < x_1 < \dots < x_n = 1\}$ произвољна подела

$\xi' = \{ \xi_k \in \mathbb{Q} : \xi_k \in (x_k, x_{k+1})\}$ $\to$ овакав скуп постоји јер је $\mathbb{Q}$ густ у $\mathbb{R}$

$\sigma(\chi, P, \xi') = \sum_{k=0}^{n-1} (x_{k+1} - x_k)\, \underbrace{\chi(\xi_k)}_{=1} = \sum_{k=0}^{n-1} (x_{k+1} - x_k) = x_n - x_0 = 1$

$\xi_k \in \mathbb{Q}$

$\xi'' = \{ \eta_k \in \mathbb{R}\setminus\mathbb{Q} : \eta_k \in (x_k, x_{k+1})\}$

$\sigma(\chi, P, \xi'') = \sum_{k=0}^{n-1} (x_{k+1} - x_k)\, \underbrace{\chi(\eta_k)}_{=0} = 0$

$\Rightarrow$ за сваку поделу имамо скупове ξ' и ξ''

$\Rightarrow \lim\limits_{\lambda(P)\to 0} \sigma(f, P, \xi)$ не постоји!

$* f \in \mathcal{R}[0,1]$, $P_n = \{ x_k : x_k = \frac{k}{n} \}$, $\lambda(P_n) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

$\xi = \{ \xi_k = x_{k+1} \}$

$$\Rightarrow \int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} (x_k - x_{k-1}) \cdot f(\xi_{k-1}) =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cdot f\left(\frac{k}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

② $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{n^2 : n^2}{k^2+n^2 : n^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\left(\frac{k}{n}\right)^2+1}$$

$$\underset{f(x)=\frac{1}{x^2+1}}{=} \int_0^1 \frac{dx}{x^2+1} = \operatorname{arctg} x \Big|_0^1 = \frac{\pi}{4}$$

③ $$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 \ln\left(1+\frac{k}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \ln\left(1+\frac{k}{n}\right) \underset{f(x)=x^2\ln(1+x)}{=}$$

$$= \int_0^1 x^2 \ln(1+x)\, dx = \begin{bmatrix} u = \ln(1+x) & \to du = \frac{1}{1+x}dx \\ dv = x^2 dx & \to v = \frac{x^3}{3} \end{bmatrix}$$

$= \dots$ за вежбу

$$(x_{k+1} - x_k) \qquad x_k$$

④ $$\lim_{n\to\infty} \frac{1}{2n^2} \sum_{k=0}^{n-1} (2k+1) \operatorname{arctg} \frac{2k+1}{2n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{2k+1}{2n} \operatorname{arctg} \frac{2k+1}{2n}$$

$x_k = \frac{2k+1}{2n}$, $x_{k+1} - x_k = \frac{2(k+1)+1}{2n} - \frac{2k+1}{2n} = \frac{1}{n}$

$$f(x) = x \operatorname{arctg} x \in R[0,1]$$

$$P = \left\{ x_k : x_k = \frac{2k+1}{2n} \right\}$$

$$\xi = \{ \xi_k = x_k \}$$

$$\lambda(P) = \frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0$$

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sigma(f, P, \xi) = \lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\overset{\shortparallel}{(x_{k+1}-x_k)}} \sum_{k=0}^{n-1} f(\underbrace{x_k}_{\overset{\shortparallel}{\xi_k}}) =$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{2k+1}{n} \operatorname{arctg} \frac{2k+1}{n}$$

$$\int_0^1 f(x)\,dx = \int_0^1 x \operatorname{arctg} x\,dx = \underbrace{---}_{\text{за вешду}}$$