① $f \in C[0,1]$, $\int_0^1 f(x)dx = \frac{1}{5}$ $\overset{?}{\Rightarrow}$ $\exists\, \theta \in (0,1)$ $f(\theta) = \theta^4$.

- - - - -

$$\frac{1}{5} = \int_0^1 x^4 dx = \int_0^1 f(x)\,dx$$

$$\int_0^1 \left(f(x) - x^4\right) dx = 0$$

$f(x) - x^4$ је непр. на $[0,1]$

$F(x) = \int_0^x (f(t) - t^4)\,dt$ $\Rightarrow$ F диф на $(0,1)$

$F(0) = \int_0^0 \cdots dt = 0$

$F(1) = \int_0^1 (f(t) - t^4)\,dt = 0$

$\Big\}$ Рол $\Rightarrow$ $\exists\, \theta \in (0,1)$ $F'(\theta) = 0 = f(\theta) - \theta^4$

② $f \in C^1[0,1]$, $\int_0^1 f(x)dx = 0$, $|f'(x)| \le 3x$, $x \in (0,1)$ $\overset{?}{\Rightarrow}$ $|f(1)| \le 1$.

$|f(1) - f(x)| = \left|\int_x^1 f'(t)\,dt\right| \le \int_x^1 3t\,dt = 3\cdot \frac{t^2}{2}\Big|_x^1 = \frac{3}{2} - \frac{3x^2}{2}$ - - - - $\to$ не видимо како даље

$0 = \int_0^1 \underbrace{f(x)}_{u}\underbrace{dx}_{dv} = \begin{bmatrix} u = f(x) & du = f'(x)\,dx \\ dv = dx & v = x \end{bmatrix} = \underbrace{xf(x)\Big|_0^1}_{f(1)} - \int_0^1 f'(x)\cdot x\,dx$

$\Rightarrow |f(1)| = \left|\int_0^1 x\cdot f'(x)\,dx\right| \le \int_0^1 |xf'(x)|dx \underset{|f'(x)| \le 3x}{\le} \int_0^1 3x^2 dx = 3\frac{x^3}{3}\Big|_0^1 = 1.$

③ $f \in C[a,b]$, $\int_a^b f(x)dx = 0$, $M = \max|f(x)|$ $\overset{?}{\Rightarrow}$ $\left|\int_a^b xf(x)\,dx\right| \le M\frac{(b-a)^2}{4}$

- - - -

$F(x) = ?$ $\underbrace{F'(x) = f(x)}$

$F(x) = \int_a^x f(t)\,dt$, $F(b) = \int_a^b f(t)dt = 0$, $F(a) = \int_a^a f(t)dt = 0$

$\int_a^b F(x)dx = \begin{bmatrix} u = F(x) & \to du = F'(x)\,dx = f(x)dx \\ dv = dx & \to v = x \end{bmatrix} = \underbrace{x\cdot F(x)\Big|_a^b}_{\underbrace{b\cdot F(b) - a\cdot F(a)}_{=0}} - \int_a^b x f(x)\,dx$

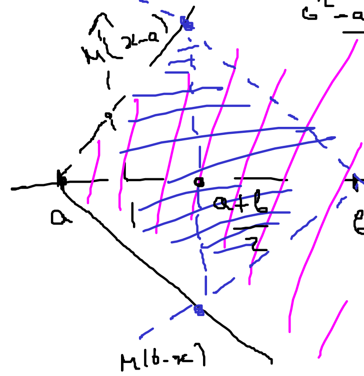
$\Rightarrow \left|\int_a^b x f(x)\,dx\right| = \left|\int_a^b F(x)\,dx\right| \le \int_a^b |F(x)|dx \le M\int_a^b (x-a)dx \le M\left(\frac{x^2}{2}\Big|_a^b - ax\Big|_a^b\right)$

$\frac{b^2 - a^2 - 2ab + 2a^2}{2}$ $=$

$|F(x) - \underset{0}{F(a)}| = \left|\int_a^x f(t)dt\right| \le \int_a^x \underbrace{|f(t)|}_{\le M}dt \le M\cdot(x-a)$

$F(b) = 0$

$|\underbrace{F(b)}_{=0} - F(x)| = |F(x)| = \left|\int_x^b f(t)\,dt\right| \le \int_x^b |f(t)|dt \le M(b-x)$



$$\left| \int_a^b x f(x)\,dx \right| = \left| \int_a^b F(x)\,dx \right| \le \int_a^b |F(x)|\,dx = \int_a^{\frac{a+b}{2}} \underbrace{|F(x)|}_{\le M(x-a)}dx + \int_{\frac{a+b}{2}}^{b} \underbrace{|F(x)|}_{M(b-x)}dx$$

$$\le \int_a^{\frac{a+b}{2}} M(x-a)\,dx + \int_{\frac{a+b}{2}}^{b} M(b-x)\,dx$$

$$= M\left(\frac{(x-a)^2}{2}\Big|_a^{\frac{a+b}{2}} - \frac{(b-x)^2}{2}\Big|_{\frac{a+b}{2}}^{b} \right) = M\left(\frac{(b-a)^2}{8} + \frac{(b-a)^2}{8} \right) = M\frac{(b-a)^2}{4}$$

(4) $f \in C^1[a,b]$, $f(a)=0 \overset{?}{\Rightarrow} \frac{2}{(b-a)^2}\int_a^b |f(x)|\,dx \le \max|f'(x)|$ $\Big\}\to$ за вежбу

$$\overset{?}{\int_a^b} |f(x)|\,dx \le \frac{(b-a)^2}{2}\max|f'(x)|$$

(5) $I_n = \int_0^1 x^n e^{\sqrt{x}}\,dx$

а) $I_0 = ?$ и веза између I_n и I_{n-1}

б) $\lim_{n\to\infty} I_n = ?$

в) $I_n = \frac{a}{n} + \frac{b}{n^2} + o\left(\frac{1}{n^2}\right)$, $a,b=?$

$$I_n = \int_0^1 x^n e^{\sqrt{x}}\,dx = \left\lceil \begin{matrix} t=\sqrt{x} \\ x=t^2 \\ dx = 2t\,dt \end{matrix} \right\rfloor = \underbrace{2\int_0^1 t^{2n+1} e^{t}\,dt}_{} = \left\lceil \begin{matrix} u = t^{2n+1} \to du = (2n+1)t^{2n}\,dt \\ dv = e^t dt \to v = e^t \end{matrix} \right\rfloor$$

$$= \underbrace{2\,t^{2n+1}\cdot e^t \Big|_0^1} - 2\cdot(2n+1)\int_0^1 t^{2n} e^t\,dt = \left\lceil \begin{matrix} u = t^{2n} \to du = 2n t^{2n-1}\,dt \\ dv = e^t dt \to v = e^t \end{matrix} \right\rfloor$$

$$= 2e - 2(2n+1)\left(t^{2n} e^t \Big|_0^1 - 2n \underbrace{\int_0^1 t^{2n-1} e^t\,dt}_{\frac{I_{n-1}}{2}} \right) =$$

$$= 2e - \overbrace{2(2n+1)}^{4n+2} e + 2n(2n+1) I_{n-1}$$

$$= -4ne + 2n(2n+1) I_{n-1}$$

$$I_0 = 2\int_0^1 t e^t\,dt = \left\lceil \begin{matrix} u = t \to du = dt \\ dv = e^t dt \to v = e^t \end{matrix} \right\rfloor = \underbrace{2te^t\Big|_0^1}_{=e} - 2\int_0^1 e^t\,dt = 2e - 2e^t\Big|_0^1 = 2e - 2e + 2$$
$$= 2$$

б) $\lim_{n\to\infty} I_n \overset{?}{=} 0$

$$I_n = \int_0^1 \underbrace{x^n e^{\sqrt{x}}}_{\overset{\vee}{0}}\,dx \quad \rightsquigarrow \quad \begin{matrix} e^{\sqrt{x}} \le e^1 \\ x^n \cdot e^{\sqrt{x}} \le x^n \cdot e \end{matrix}, \quad \int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1},$$

$0 \le I_n \le ?$

$$I_n = \int_0^1 \underbrace{x^n e^{\sqrt{x}}}_{\le x^n \cdot e} dx \le \int_0^1 x^n \cdot e \, dx \overset{\smile}{=} \frac{e}{n+1} \xrightarrow{n\to\infty} 0$$

$$0 \le I_n \le \frac{e}{n+1} \xrightarrow{n\to\infty} 0$$

$$\downarrow_{n\to\infty} 0$$

$$\lim_{n\to\infty} I_n = 0$$

б) $I_n \overset{?}{=} \frac{a}{n} + \frac{b}{n^2} + \underbrace{o\left(\frac{1}{n^2}\right)}_{= \alpha(n)\cdot\frac{1}{n^2},\ \alpha(n)\to 0}$; $a, b = ?$

$$I_n = -4ne + 2n(2n+1) I_{n-1}$$

$$I_{n+1} = -4(n+1)e + 2(n+1)(2(n+1)+1) I_n$$

$$= -4(n+1)e + (2n+2)(2n+3) I_n \longrightarrow I_n = \frac{\overset{\to 0}{I_{n+1}} + 4(n+1)e}{(2n+2)(2n+3)}$$

$$\lim_{n\to\infty} I_n = 0 \Rightarrow \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ n \ge n_0 \ |I_n| \le \varepsilon$$

$$I_n = \underbrace{\frac{I_{n+1}}{(2n+2)(2n+3)}}_{= o\left(\frac{1}{n^2}\right)} + \frac{\overset{2}{\cancel{4}}(n+1)e}{(\cancel{2n+2})(2n+3)} = \frac{2e}{2n+3} + o\left(\frac{1}{n^2}\right) =$$

$$= 2e(2n+3)^{-1} + o\left(\frac{1}{n^2}\right) = \frac{\cancel{2}e}{\cancel{2}n}\underbrace{\left(1+\frac{3}{2n}\right)^{-1}}_{\left(1-\frac{3}{2n}+o\left(\frac{1}{n}\right)\right)} + o\left(\frac{1}{n^2}\right) = \frac{e}{n} - \frac{3e}{2n^2} + \underbrace{o\left(\frac{1}{n^2}\right) + o\left(\frac{1}{n^2}\right)}_{o\left(\frac{1}{n^2}\right)}$$

$a = e$, $b = -\frac{3e}{2}$.

6) $F : (-1, +\infty) \to \mathbb{R}$, $F(x) = \int_{-1/2}^{x} \frac{t + \operatorname{arctg} t}{1+t}\, dt$

Испитати ток ф-је F (монотоност, нуле и показати да постоји бар једна превојна тачка).

$$F\left(-\frac{1}{2}\right) = \int_{-1/2}^{-1/2} \dots dt = 0$$

$$F'(x) = \frac{x + \operatorname{arctg} x}{1+x} \quad x \in (-1, +\infty)$$

$x + \operatorname{arctg} x > 0$, $x > 0$

$x + \operatorname{arctg} x < 0$, $x < 0$

$x + \operatorname{arctg} x = 0$, $x = 0$

	-1		$-1/2$		0	
F'		$-$		$-$		$+$
F		$\searrow$		$\searrow$		$\nearrow$

локални минимум

$F(x) > F(-\frac{1}{2}) = 0 \quad , \; x > -\frac{1}{2}$

На $(-1, -\frac{1}{2})$ → немамо нуле

$F(0) < F(-\frac{1}{2}) = 0$

$x \to +\infty, \; F(x) = \int_{-1/2}^{x} \frac{t + \operatorname{arctg} t}{1+t} dt = \underbrace{\int_{-1/2}^{0} \frac{t+\operatorname{arctg} t}{1+t} dt}_{F(0) < 0} + \underbrace{\int_{0}^{x} \frac{t+\operatorname{arctg} t}{1+t} dt}_{> 0}$

$\frac{t + \operatorname{arctg} t}{1+t} > 1$

$t > \sqrt{3}$

$\operatorname{arctg} t > \frac{\pi}{3} > 1$

$= F(0) + \underbrace{\int_{0}^{\sqrt{3}} \frac{t+\operatorname{arctg} t}{1+t} dt}_{c} + \int_{\sqrt{3}}^{x} \overbrace{\frac{t+\operatorname{arctg} t}{1+t}}^{>1} dt$

$> F(0) + c + \int_{\sqrt{3}}^{x} dt = F(0) + c + x - \sqrt{3} \xrightarrow{x \to +\infty} +\infty$

$\Rightarrow F(x) \to +\infty, \; x \to +\infty$

$\Rightarrow F(0) < 0, \; F(x) \to +\infty, x \to +\infty \Rightarrow \exists \, a \in (0, +\infty) \; F(a) = 0.$

превојна тачка:

$F''(x) = \left(\frac{x + \operatorname{arctg} x}{1+x}\right)' = \frac{(1+x)(1+\frac{1}{1+x^2}) - (x + \operatorname{arctg} x)}{(1+x)^2}$

$= \frac{1 + \frac{1+x}{1+x^2} - \operatorname{arctg} x}{\underbrace{(1+x)^2}_{>0}}$

$f(x) = 1 + \frac{1+x}{1+x^2} - \operatorname{arctg} x$

$f'(x) = \frac{1+x^2 - 2x(1+x)}{(1+x^2)^2} - \frac{1}{1+x^2} = \frac{\cancel{1}+\cancel{x^2} - 2x(1+x) - (\cancel{1}+\cancel{x^2})}{(1+x^2)^2} = \frac{-2x(1+x)}{(1+x^2)^2}, \; x > -1$

	-1		0	
f'		$+$		$-$
f		$\nearrow$		$\searrow$

$f(0) = 2 \quad , \; f(-1) = 1 + \frac{1-1}{2} - \operatorname{arctg}(-1) = 1 + \frac{\pi}{4} > 0$

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} 1 + \frac{1+x}{1+x^2} - \operatorname{arctg} x = 1 - \frac{\pi}{2} < 0 \quad , f(0) > 2 \Rightarrow f$ мења знак

$\Rightarrow \exists$ тачно једна превојна тачка фје F.

* Коши-Шварцова интегрална неједнакост

$f, g \in \mathcal{R}[a,b], \; a < b \Rightarrow \left(\int_a^b f(x) g(x) dx\right)^2 \leq \int_a^b f(x)^2 dx \cdot \int_a^b g(x)^2 dx$, "=" акко $f(x) = \lambda g(x)$

$[a_1, \dots, a_n, b_1, \dots, b_n$

$(a_1 b_1 + \dots a_n b_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)$, "=" акко $a_k = \lambda b_k$

(7) $f \in C^1[0,1]$, $f(1)-f(0)=1 \overset{?}{\Rightarrow} \int_0^1 f'(x)^2\,dx \geq 1$

$$1^2 = |f(1)-f(0)|^2 = \left(\int_0^1 1\cdot f'(x)\,dx\right)^2 \underset{C-B}{\leq} \int_0^1 1^2\,dx \cdot \int_0^1 f'(x)^2\,dx$$

$$(f')^2 \geq 2f'-1 \quad / \int_0^1 \quad \Rightarrow \quad \int_0^1 (f'(x))^2\,dx \geq \underbrace{\int_0^1 2f'(x)\,dx}_{2(f(1)-f(0))} - \underbrace{\int_0^1 dx}_{1} = 1.$$

$$(f')^2 - 2f' + 1 \geq 0$$

(8)

$$A = \left\{ f \in C^1[0,1] : f(0)=0,\ \int_0^1 f'(x)^2\,dx \leq 1 \right\}$$

$$\sup_{f\in A} \int_0^1 |f'(x)|^2\,|f(x)|\,\frac{1}{\sqrt{x}}\,dx = ?$$

$$\int_0^1 |f'(x)|^2\,|f(x)|\,\frac{1}{\sqrt{x}}\,dx = ?$$

$f'(t) = \lambda \cdot 1$ $\quad 1 = \int_0^1 f'(x)^2\,dx = \lambda^2\cdot 1 \Rightarrow \lambda = 1$

$\Rightarrow f(x) = x$, $x\in[0,1]$

$$(f(x))^2 = |f(x) - \underbrace{f(0)}_{=0}|^2 = \left|\int_0^x f'(t)\,dt\right|^2 \leq \int_0^x f'(t)^2\,dt \cdot \int_0^x 1\,dt = \int_0^x f'(t)^2\,dt \cdot x$$

$$|f(x)| \leq \left(\int_0^x f'(t)^2\,dt\right)^{1/2}\cdot\sqrt{x} \leq \left(\int_0^1 f'(t)^2\,dt\right)^{1/2}\sqrt{x}$$

$$\Rightarrow \int_0^1 |f'(x)|^2\cdot|f(x)|\cdot\frac{1}{\sqrt{x}}\,dx \leq \int_0^1 f'(x)^2\cdot\sqrt{\int_0^x f'(t)^2\,dt}\cdot\cancel{\sqrt{x}}\cdot\frac{1}{\cancel{\sqrt{x}}}\,dx \quad \text{d4}$$

$$= \quad u = \int_0^x f'(t)^2\,dt = \varphi(x) \quad = \int_0^{u(1)} \sqrt{u}\,du = \frac{u^{3/2}}{3/2}\Big|_0^{u(1)} = \frac{u(1)^{3/2}}{\frac{3}{2}} \leq \frac{2}{3}$$

$du = f'(x)^2\,dx$

$\varphi'(x) = f'(x)^2 \geq 0 \Rightarrow \varphi \nearrow$

$u(0) = \int_0^0 f'(x)^2\,dx = 0$

$u(1) = \int_0^1 f'(x)^2\,dx \leq 1$

$f(x) = x$, $f\in A$? $\quad f(0)=0$ ✓ $\quad \int_0^1 f'(x)^2\,dx = 1$ ✓

$$\int_0^1 \underbrace{f'(x)^2}_{=1}\cdot\underbrace{|f(x)|}_{=x}\cdot\frac{1}{\sqrt{x}}\,dx = \int_0^1 \sqrt{x}\,dx = \frac{2}{3} \ ✓$$

$$\sup_A \cdots = \max_A \cdots = \frac{2}{3}$$

$\downarrow$

$f(x) = x$.