* $F(x)=\int_a^x f(t)\,dt$, $f\in \mathcal{R}[a,b]$

$f\in C[a,b] \Rightarrow F\in C^1[a,b]$, $F'(x)=f(x)$

(1) $f\in C(\mathbb{R})$ периодична са периодом $T>0$, $f(x+T)=f(x)$ $\forall x\in\mathbb{R}$

$\overset{?}{\Rightarrow} \forall a\in\mathbb{R}$ $\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$.

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$F(a)=\int_a^{a+T}f(x)\,dx$, $a\in\mathbb{R}$

$F(a)=F(0)$?

извод је по a

$$F'(a) = \left(\int_a^{a+T} f(x)\,dx\right)' = \left(\int_0^{a+T} f(x)\,dx + \int_a^0 f(x)\,dx\right)'$$

$$= \left(\int_0^{a+T}f(x)\,dx\right)' - \left(\int_0^a f(x)\,dx\right)'$$

овде користимо непр. ф-је f

$$= \underbrace{f(a+T)}_{f(a)}\cdot\underbrace{(a+T)'}_{=1} - f(a) = 0$$

$\Rightarrow F(a)=const.=F(0)=\int_0^T f(x)\,dx$.

$\{x\}=x-[x]$ → пример овакве ф-је

(2) $f\in\mathcal{R}[a,b]$, за свако $[a,b]\subseteq\mathbb{R}$ и периодична са периодом T

$\overset{?}{\Rightarrow}\forall a\in\mathbb{R}$ $\int_a^{a+T}f(x)\,dx=\int_0^T f(x)\,dx$

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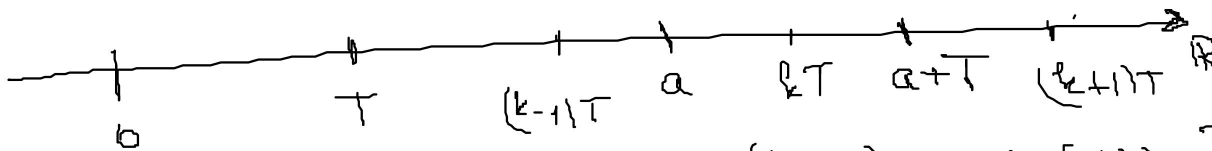


$t=x-a$ → $f(x)=f(t+a)$ → $f(t)$?

$\int_a^{a+T}f(x)\,dx=$

$t = x - \underbrace{kT}$

$f(x) = f(t + \underbrace{kT}) = f(t)$

$\exists k \in \mathbb{Z} \qquad (k-1)T < a \leq kT < a+T < (k+1)T$

$$\int_a^{a+T} f(x)\,dx = \int_a^{kT} f(x)\,dx + \underbrace{\int_{kT}^{a+T} f(x)\,dx} = \int_a^{kT} f(x)\,dx + \int_{(k-1)T}^{a} f(u)\,du \quad u = x$$

$u = x - T$

$u(a+T) = a$

$u(kT) = (k-1)T$

$du = dx$

$f(x) = f(u+T) = f(u)$ f периог.

$$= \int_{(k-1)T}^{kT} f(x)\,dx = \qquad \int_0^T f(w)\,dw \underset{w=x}{=} \int_0^T f(x)\,dx$$

$w = x - (k-1)T$

$w(kT) = T$

$w((k-1)T) = 0$

$f(x) = f(w + (k-1)T) = f(w)$

$dw = dx$

(3) $f : \mathbb{R} \to \mathbb{R}$ T-периодична, непрекидна и непарна на $\left[-\frac{T}{2}, \frac{T}{2}\right]$

$\overset{?}{\Rightarrow} \quad F(x) = \int_0^x f(t)\,dt \quad$ T-периодична ф-ја

$? \; F(x+T) = F(x), \quad x \in \mathbb{R}$

$? \int_0^{x+T} f(t)\,dt = \int_0^x f(t)\,dt \iff ? \int_0^{x+T} f(t)\,dt - \int_0^x f(t)\,dt = 0$

$$\int_0^{x+T} f(t)\,dt + \int_x^0 f(t)\,dt = \int_x^{x+T} f(t)\,dt = \int_0^T f(t)\,dt = \int_{a+0}^{a+T} f(t)\,dt$$

$\downarrow$ $a \in \mathbb{R}$

$$\underset{a=-\frac{T}{2}}{=} \int_{-T/2}^{T/2} f(t)\,dt \underset{f \text{ непарно}}{=} 0$$

$*$ $f, g \in R[a,b]$, $f(x) \le g(x) \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

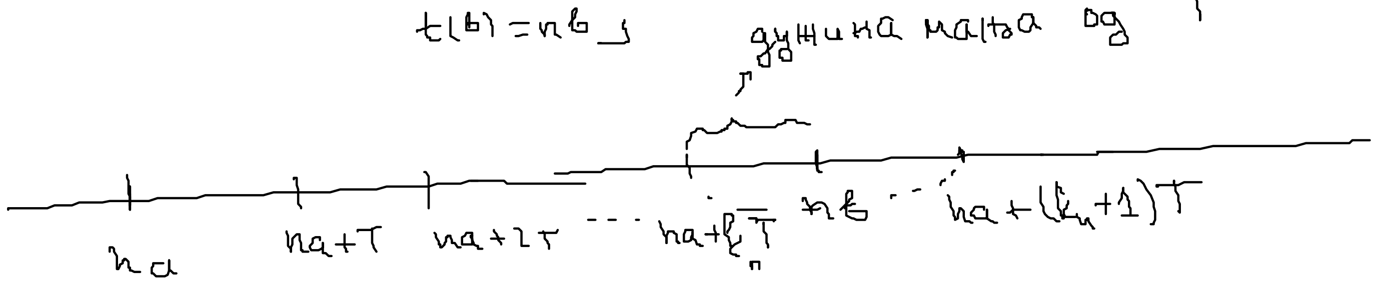
$f=0 \Rightarrow 0 \le g(x) \Rightarrow 0 \le \int_a^b g(x)\,dx$

$a<b$ $\quad \left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx \le \sup_{a \le x \le b} |f(x)| \cdot \underbrace{\int_a^b dx}_{b-a}$

(4) f непрекидна на $\mathbb{R}$, T-период.

$$\overset{?}{\Rightarrow} \lim_{n\to\infty} \int_a^b f(nx)\,dx = \frac{b-a}{T} \int_0^T f(x)\,dx$$

$$\int_a^b f(nx)\,dx = \left[\begin{array}{l} t = nx \\ f(nx) = f(t) \\ dx = \frac{dt}{n} \\ t(a) = na \\ t(b) = nb \end{array}\right. = \frac{1}{n} \int_{na}^{nb} f(t)\,dt = (*)$$



$$na + \underline{k_n}T \le nb < na + (k_n+1)T$$

$k_nT \le nb - na$ $\qquad\qquad$ $k_nT + T > nb - na$

$k_n \le \frac{n(b-a)}{T}$ $\qquad\qquad$ $k_n > \frac{nb-na}{T} - 1$

$$\circledast = \frac{1}{n}\left(\underbrace{\int_{na}^{na+T} f(t)\,dt}_{\int_0^T f(t)dt} + \underbrace{\int_{na+T}^{na+2T} f(t)\,dt}_{\int_0^T f(t)dt} + \dots + \underbrace{\int_{na+(k_n-1)T}^{na+k_nT} f(t)\,dt}_{\int_0^T f(t)dt} + \int_{na+k_nT}^{nb} f(t)\,dt \right) =$$

$$= \frac{k_n}{n}\int_0^T f(t)\,dt + \underbrace{\frac{1}{n}\int_{na+k_nT}^{nb} f(t)\,dt}$$

$$\frac{k_n}{n} \xrightarrow{n\to\infty} ? \qquad \frac{1}{n}\int_{na+k_nT}^{nb} f(t)\,dt \xrightarrow{n\to\infty} ?$$

$$\frac{\frac{n(b-a)}{T}-1}{n} < \frac{k_n}{n} \le \frac{\frac{n(b-a)}{T}}{n}$$

$$\frac{b-a}{T} - \frac{1}{n} < \frac{k_n}{n} \le \frac{b-a}{T}$$

$$\frac{b-a}{T}-\frac{1}{n} \xrightarrow{n\to\infty} \frac{b-a}{T}, \qquad \frac{k_n}{n} \xrightarrow{\text{Т о 2 п}} \frac{b-a}{T}, \qquad \frac{b-a}{T} \to \frac{b-a}{T}$$

$$0 \le \left| \frac{1}{n}\int_{na+k_nT}^{nb} f(t)\,dt \right| \le \frac{1}{n}\int_{na+k_nT}^{nb} |f(t)|\,dt \le \frac{M}{n}\int_{na+k_nT}^{nb} dt$$

$$= \frac{M}{n}\cdot\big(\underbrace{nb-(na+k_nT)}_{<T}\big) < \frac{M\cdot T}{n} \xrightarrow{n\to\infty} 0$$

на основу конструкције k_n

$f \in C(\mathbb{R})$ $\quad T$ - период

f непр. на $[0,T]$ $\overset{\text{Вајерштрас}}{\Longrightarrow}$ f огран на $[0,T]$ са M

$|f(x)| \le M$

$\Rightarrow |f(x)| \le M \quad \forall x \in \mathbb{R}$

$$\frac{1}{n}\int_{na+k_nT}^{nb} f(t)\,dt \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \lim_{n\to\infty} \int_a^b f(nx)\,dx = \frac{b-a}{T}\int_0^T f(x)\,dx$$

(5) $f \in C[1,8]$: $\displaystyle\int_1^2 f^2(t^3)\,dt + 2\int_1^2 f(t^3)\,dt = \frac{2}{3}\underbrace{\int_1^8 f(t)\,dt}_{} - \int_1^2 (t^2-1)^2\,dt$

$f = ?$

$u = \sqrt[3]{t}$

$u(8) = 2$

$u(1) = 1$

$t = u^3$

$dt = 3u^2\,du$

$$\Rightarrow \int_1^2 f^2(t^3)\,dt + 2\int_1^2 f(t^3)\,dt = \frac{2}{\cancel{3}}\underbrace{\int_1^2 f(u^3)\cdot \cancel{3}u^2\,du}_{u=t} - \int_1^2 (t^2-1)^2\,dt$$

$$\int_1^2 \Big(f^2(t^3) + \underbrace{2f(t^3) - 2f(t^3)\cdot t^2}_{-2f(t^3)(t^2-1)} + (t^2-1)^2 \Big)\,dt = 0$$

$$\Rightarrow \int_1^2 \underbrace{\big(f(t^3) - (t^2-1)^2 \big)^2}_{\geq 0}\,dt = 0 \quad (*)$$

☺ $g(x)$ непр на $[a,b]$, $g(x) \geq 0$ и $\int_a^b g(x)\,dx = 0 \Rightarrow g(x) = 0$, $x \in [a,b]$

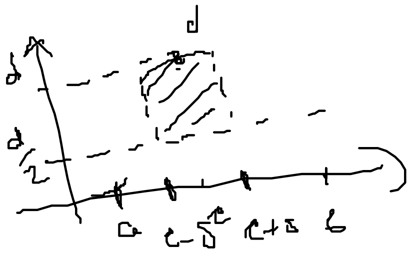
Δ: ПП. $\exists\, c \in [a,b]$ $g(c) = d > 0$

$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in [a,b] \;\; |x - c| < \delta \Rightarrow |g(x) - \underset{g(c)}{d}| < \varepsilon$

$g(x) \in (d-\varepsilon, d+\varepsilon)$

$\varepsilon = \frac{d}{2} > 0 \Rightarrow \delta(\frac{d}{2}) > 0$ и $|x - c| < \delta \Rightarrow g(x) > \frac{d}{2}$

$(\frac{d}{2}, \frac{3d}{2})$



$$\int_a^b g(x)\,dx = \underbrace{\int_a^{c-\delta} \underbrace{g(x)}_{\geq 0}\,dx}_{\geq 0} + \underbrace{\int_{c-\delta}^{c+\delta} \underbrace{g(x)}_{\geq \frac{d}{2}}\,dx}_{\geq \frac{d}{2}\cdot 2\delta} + \underbrace{\int_{c-\delta}^{b} \underbrace{g(x)}_{\geq 0}\,dx}_{\geq 0} > 0$$

$\lightning \Rightarrow g(x) = 0$

:) $u(*) \Rightarrow \left(f(t^3) - (t^2-1)^2\right)^2 = 0, \quad t\in[1,2]$

обде користимо нер. віе

$f(t^3) = (t^2-1)^2, \quad t\in[1,2]$

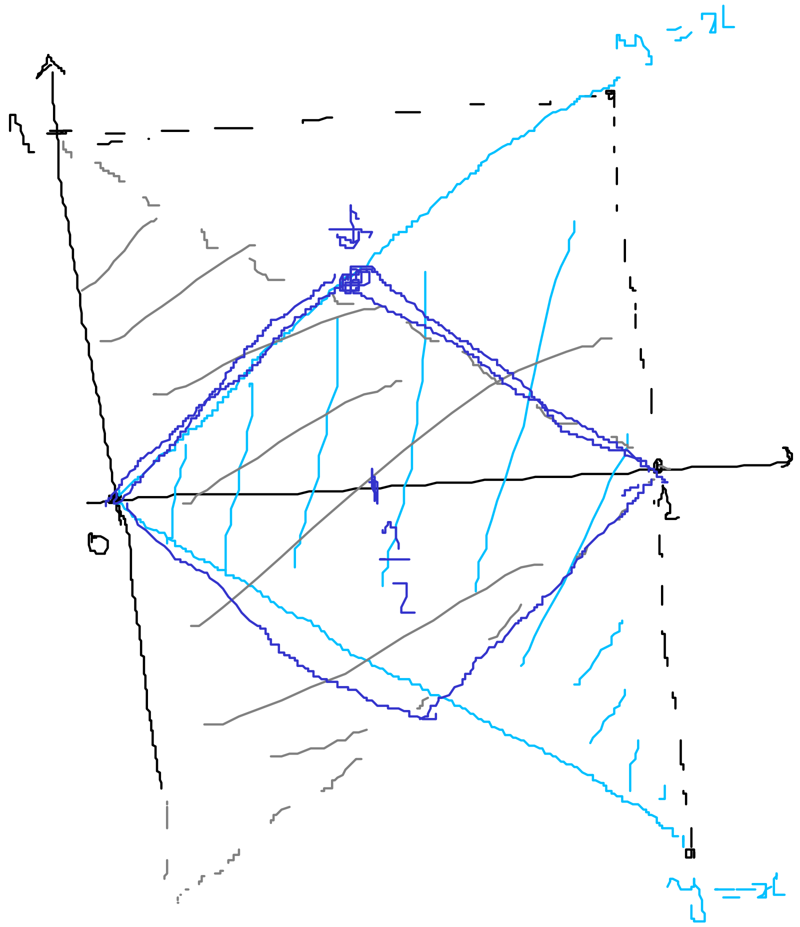
$x = t^3$

$t = x^{1/3} \longrightarrow \boxed{f(x) = (x^{2/3}-1)^2, \quad x\in[1,8]}$

$t\in[1,2] \Rightarrow x\in[1,8]$

6) $f\in C[0,1]\cap D(0,1), \quad f(0)=f(1)=0, \quad |f'(x)|\le 1 \quad x\in(0,1)$

$\overset{?}{\Rightarrow} \left|\int_0^1 f(x)\,dx\right| \le \frac{1}{4}$



$f(x) - \underset{\overset{\shortparallel}{0}}{f(0)} = \int_0^x f'(t)\,dt$

$f(x) = \int_0^x f'(t)\,dt$

$|f(x)| = \left|\int_0^x f'(t)\,dt\right| \le \int_0^x \underbrace{|f'(t)|}_{\le 1}\,dt$

$\le 1\cdot\int_0^x dt = x$

$\Rightarrow |f(x)| \le x, \quad x\in[0,1]$

$\left|\int_0^1 f(x)\,dx\right| \le \int_0^1 \underbrace{|f(x)|}_{\le x}\,dx \le \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

$|-f(x)| = |f(1)-f(x)| = \left|\int_x^1 f'(t)\,dt\right| \le \int_x^1 |f'(t)|\,dt \le \int_x^1 dt = 1-x$

$|f(x)| \le 1-x$

$$\left|\int_0^1 f(x)\,dx\right| = \left|\int_0^{1/2} f(x)\,dx + \int_{1/2}^1 f(x)\,dx\right| \leq$$

$$\leq \left|\int_0^{1/2} f(x)\,dx\right| + \left|\int_{1/2}^1 f(x)\,dx\right|$$

$$\leq \int_0^{1/2} \underbrace{|f(x)|}_{\leq x}\,dx + \int_{1/2}^1 \underbrace{|f(x)|}_{\leq 1-x}\,dx$$

$$\leq \int_0^{1/2} x\,dx + \int_{1/2}^1 (1-x)\,dx = \frac{x^2}{2}\Big|_0^{1/2} + x\Big|_{1/2}^1 - \frac{x^2}{2}\Big|_{1/2}^1$$

$$= \frac{1}{8} + \frac{1}{2} - \frac{1}{2} + \frac{1}{8} = \frac{1}{4}$$