* $$\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \underbrace{\int_{-a}^{0} f(x)\,dx}_{\substack{t=-x \\ dt=-dx}} = \int_0^a f(x)\,dx + \underbrace{\int_a^0 f(-t)(-dt)}_{x=t}$$

$$= \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx = \int_0^a \big(f(x)+f(-x)\big)\,dx$$

f непарна $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

f парна $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(1) $$\int_{-\pi/2}^{\pi/2} \frac{\sin^3 x + 2\sin x + \cos^3 x + 2\cos x + \overbrace{x}^{\text{непарно}}\cdot\overbrace{\ln(\cos^{2021}x+17)}^{\text{парно}}}{\underbrace{(\cos^2 x+2)(\cos^2 x+1)}_{g(x)\ \to\ \text{парна}}}\,dx =$$

$$= \underbrace{\int_{-\pi/2}^{\pi/2} \frac{\sin^3 x}{g(x)}\,dx}_{\substack{\text{непарна} \\ 0}} + \underbrace{\int_{-\pi/2}^{\pi/2} \frac{2\sin x}{g(x)}\,dx}_{\substack{\text{непарна} \\ =0}} + \int_{-\pi/2}^{\pi/2} \underbrace{\frac{\cos^3 x}{g(x)}}_{\text{парна}}\,dx + \int_{-\pi/2}^{\pi/2} \underbrace{\frac{2\cos x}{g(x)}}_{\text{парна}}\,dx + \underbrace{\int_{-\pi/2}^{\pi/2} \frac{x\ln(\cos^{2021}x+17)}{g(x)}\,dx}_{\substack{\text{непарно} \\ =0}}$$

f – парна $\Rightarrow F = \int$ парна

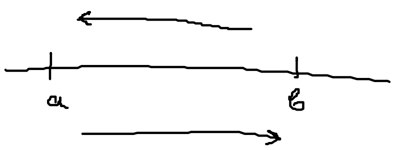
F → [illegible] ф-ія

$$= 2\int_0^{\pi/2} \frac{\cos^3 x + 2\cos x}{(\cos^2 x+2)(\cos^2 x+1)}\,dx = 2\int_0^{\pi/2} \frac{\cos x\,\cancel{(\cos^2 x+2)}}{\cancel{(\cos^2 x+2)}(\cos^2 x+1)}\,dx = \left[\begin{array}{ll} t=\sin x, & t(0)=0 \\ & t(\tfrac{\pi}{2})=1 \\ dt=\cos x\,dx & \\ \cos^2 x = 1-t^2 & \end{array}\right]$$

$$= 2\int_0^1 \frac{dt}{\underbrace{2-t^2}} = \frac{1}{\sqrt2}\ln\left(\frac{\sqrt2+t}{\sqrt2-t}\right)\Bigg|_0^1 = \boxed{\frac{1}{\sqrt2}\ln\frac{\sqrt2+1}{\sqrt2-1}} - \underbrace{\frac{1}{\sqrt2}\ln 1}_{=0}$$

$$\frac{1}{2\sqrt2}\left(\frac{1}{\sqrt2-t} + \frac{1}{\sqrt2+t}\right)$$

* $$\int_a^b f(x)\,dx = \left[\begin{array}{l} x=b+a-t \\ t=b+a-x \\ dt=-dx \end{array}\right] = \int_b^a f(b+a-t)(-dt) = \int_a^b f(b+a-t)\,dt \underset{x=t}{=} \int_a^b f(b+a-x)\,dx$$



$$\left.\begin{array}{l} t(a)\to b \\ t(b)\to a \end{array}\right\}\quad t=b+a-x$$

$t = kx + n$

$ka + n = b$

$kb + n = a$

$k(a-b) = b-a \Rightarrow k=-1$

$n = b+a$

(2) $$\int_{-a}^{a} x^2\frac{e^x}{1+e^x}\,dx = \left[\begin{array}{l} x=-t \\ dx=-dt \end{array}\right] = \int_a^{-a} t^2\frac{e^{-t}}{1+e^{-t}}\cdot\frac{e^t}{e^t}(-dt) = \int_{-a}^{a} t^2\frac{1}{e^t+1}\,dt = \int_{-a}^{a} x^2\frac{dx}{e^x+1} = I$$

$$2I = \int_{-a}^{a} x^2\frac{e^x}{1+e^x}\,dx + \int_{-a}^{a} x^2\frac{dx}{1+e^x} = \int_{-a}^{a} x^2\frac{e^x+1}{1+e^x}\,dx = \int_{-a}^{a} x^2\,dx = \frac{x^3}{3}\Big|_{-a}^{a} = \frac{2a^3}{3}$$

$$\Rightarrow \boxed{I = \frac{a^3}{3}}$$

* f непрерывна

$$(1)\ \int_0^{\pi/2} f(\sin x)\,dx = \left[x = \frac{\pi}{2} - t\right] = \int_0^{\pi/2} f\left(\sin\left(\frac{\pi}{2}-t\right)\right)dt = \int_0^{\pi/2} f(\cos t)\,dt = \int_0^{\pi/2} f(\cos x)\,dx$$

$$(2)\ \underbrace{\int_0^{\pi} x\, f(\sin x)\,dx}_{I} = \left[\begin{array}{l} x = \pi - t \\ dx = -dt \end{array}\right] = \int_0^{\pi} (\pi - t)\, f(\underbrace{\sin(\pi - t)}_{\sin t})\,dt = \underbrace{\int_0^{\pi} (\pi - x) f(\sin x)\,dx}_{\int_0^{\pi} \pi f(\sin x)\,dx - I}$$

$$\Rightarrow\ 2I = \pi \int_0^{\pi} f(\sin x)\,dx$$

$$\boxed{\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx}$$

$$(3)\ \int_0^{\pi} \frac{x\sin x}{1+\underbrace{\cos^2 x}_{1-\sin^2 x}}\,dx \overset{(2)}{=} \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{2-\sin^2 x}\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = \left[\begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \\ t(0) = 1 \\ t(\pi) = -1 \end{array}\right]$$

$$f(\sin x) = \frac{\sin x}{2-\sin^2 x}$$

$$= -\frac{\pi}{2}\int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{2}\,\mathrm{arctg}\, t\Big|_{-1}^{1} = \frac{\pi^2}{4}$$

$$(4)\ \underbrace{\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx}_{I} = \left[\begin{array}{l} t = \frac{\pi}{2} - x \\ dt = -dx \end{array}\right] = \int_{\pi/2}^{0} \frac{\left(\frac{\pi}{2}-t\right)\overbrace{\sin\left(\frac{\pi}{2}-t\right)}^{=\cos t}\overbrace{\cos\left(\frac{\pi}{2}-t\right)}^{=\sin t}}{\underbrace{\sin^4\left(\frac{\pi}{2}-t\right)}_{\cos^4 t} + \underbrace{\cos^4\left(\frac{\pi}{2}-t\right)}_{\sin^4 t}}(-dt) \underset{t=x}{=}$$

$$= \int_0^{\pi/2} \frac{\pi}{2}\cdot\frac{\cos x\sin x}{\cos^4 x + \sin^4 x} - \underbrace{\int_0^{\pi/2} \frac{x\cos x\sin x}{\cos^4 x+\sin^4 x}\,dx}_{I}$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\overbrace{\cos x\sin x}^{=\frac{1}{2}\sin 2x}}{\underbrace{\cos^4 x + \sin^4 x}_{(\cos^2 x + \sin^2 x)^2 - \underbrace{2\sin^2 x\cos^2 x}_{\frac{1}{2}\sin^2 2x}}}\,dx =$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\frac{1}{2}\sin 2x\,dx}{1-\frac{1}{2}\sin^2 2x} = \left[\begin{array}{l} t = \cos 2x \\ dt = -\sin 2x\cdot 2dx \end{array}\right] = \frac{\pi}{2}\int_1^{-1} \frac{-\frac{1}{4}\,dt}{\frac{1}{2}+\frac{1}{2}t^2} = \frac{\pi}{4}\int_{-1}^{1}\frac{dt}{1+t^2} =$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

$$\Rightarrow \boxed{I = \frac{\pi^2}{16}}$$

$$(5)\ \underbrace{\int_{1/a}^{a} \frac{\ln x}{1+x^2}\,dx}_{I} = \left[\begin{array}{l} t = \frac{1}{x},\quad x = \frac{1}{t} \\ dx = -\frac{dt}{t^2} \end{array}\right] = \int_a^{1/a} \frac{\overbrace{\ln\frac{1}{t}}^{=-\ln t}}{1+\frac{1}{t^2}}\left(-\frac{dt}{t^2}\right) = -\underbrace{\int_{1/a}^{a} \frac{\ln t\,dt}{1+t^2}}_{-I}$$

$2I = 0 \Rightarrow I = 0.$

$a > 0$

(6) $$\int_{1/a}^{a} \frac{|\ln x|}{1+x}\,dx = \int_{1/a}^{1} \frac{|\ln x|}{1+x}\,dx + \int_{1}^{a} \frac{|\ln x|}{1+x}\,dx$$

1° $a > 1$, $\frac{1}{a} < 1$

$$\int \frac{\ln x}{1+x}\,dx = \begin{bmatrix} u = \ln x & du = \frac{1}{x}dx \\ dv = \frac{dx}{1+x} & \to v = \ln(1+x) \end{bmatrix} = \ln x \ln(1+x) - \int \frac{\ln(1+x)\,dx}{x}$$

не може овако ...

...

II начин

$$\underbrace{\int_{1/a}^{a} \frac{|\ln x|}{1+x}\,dx}_{I} = \begin{bmatrix} t = \frac{1}{x} & x = \frac{1}{t} \\ dx = -\frac{dt}{t^2} & \end{bmatrix} = \int_{a}^{1/a} \frac{\overset{|\ln t|}{\left|\ln \frac{1}{t}\right|}}{\underset{\frac{x+1}{x}}{1 + \frac{1}{t}}} \cdot \left(-\frac{dt}{t^2}\right) = \underbrace{\int_{1/a}^{a} \frac{|\ln t|}{t\cdot(1+t)}\,dt}_{I} \xrightarrow{t=x}$$

$$2I = \int_{1/a}^{a} \frac{|\ln x|}{1+x}\left(1 + \frac{1}{x}\right)dx = \int_{1/a}^{a} \frac{|\ln x|}{x}\,dx = \begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{bmatrix} = \int_{-\ln a}^{\ln a} |u|\,du$$

1° $a < 1$

$$2I = \int_{-\ln a}^{0} \underbrace{|u|}_{u}du + \int_{0}^{\ln a} \underbrace{|u|}_{-u}du = \frac{u^2}{2}\Big|_{-\ln a}^{0} - \frac{u^2}{2}\Big|_{0}^{\ln a} = -\frac{\ln^2 a}{2} - \frac{\ln^2 a}{2} = -\ln^2 a$$

$$I = -\frac{\ln^2 a}{2}$$

2° $a > 1$

$$I = \frac{\ln^2 a}{2}$$

3° $a = 1$ $\longrightarrow$ $2I = \int_0^0 |u|\,du = 0$ $\quad \left[\int_a^a f(x)\,dx = 0\right]$

$*$ $f \in R[a,b]$ $\quad F(x) = \int_a^x f(t)\,dt$, $x \in [a,b] \Rightarrow F \in C[a,b]$

f непрекидна у x_0 $\Rightarrow F$ диференцијабилна у x_0 и $F'(x_0) = f(x_0)$

$f \in C[a,b] \Rightarrow \left(\int_a^x f(t)\,dt\right)'_{\text{по } x} = f(x)$ извод

$$\left(\int_a^{g(x)} f(t)\,dt\right)'_{\text{по } x} = f(g(x))\cdot g'(x) \Leftarrow$$

① $F(x) = \int_{\sqrt{x}}^{2\sqrt{x}} e^{-t^2}\,dt \quad , \longrightarrow F'(x)$

$$= \int_{\sqrt{x}}^{0} e^{-t^2}dt + \int_{0}^{2\sqrt{x}} e^{-t^2}dx = -\overbrace{\int_{0}^{\sqrt{x}} e^{-t^2}dt}^{=g(x)} + \int_{0}^{2\sqrt{x}} e^{-t^2}dt$$

$$F'(x) = -e^{-(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}} + e^{-(2\sqrt{x})^2}\cdot\frac{1}{\sqrt{x}}$$

$$= -e^{-x}\cdot\frac{1}{2\sqrt{x}} + e^{-4x}\cdot\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}}\left(e^{-4x} - \frac{1}{2}e^{-x}\right) \quad . . .$$

② $f \in C[0,\infty]$, $\lim\limits_{x\to+\infty} f(x) = 2022 > 0 \Rightarrow \int_{22}^{x} f(t)dt \underset{x\to+\infty}{\longrightarrow} +\infty$ и $x \to +\infty$

$$\lim_{x\to+\infty} \frac{\int_{22}^{x} f(t)dt}{x} \overset{ЛП}{=} \lim_{x\to+\infty} \frac{\left(\int_{22}^{x} f(t)dt\right)'}{(x)'} = \lim_{x\to+\infty}\frac{f(x)}{1} = 2022$$

$\int_{22}^{x} f(t)dt \to$ диференцијабилно по x

$$\int_{22}^{x} f(t)dt \sim 2022\cdot x + \text{const}$$

③ $$\lim_{x\to+\infty} \frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt} = \left\lceil \int_0^x e^{t^2}dt \underset{x\to+\infty}{\longrightarrow} +\infty \text{ јер } e^{x^2} \underset{x\to+\infty}{\longrightarrow} +\infty > 0 \right\rfloor \overset{ЛП}{=}$$

$$\lim_{x\to+\infty} \frac{2\int_0^x e^{t^2}dt \cdot \cancel{e^{x^2}}}{e^{2x^2}} \overset{ЛП}{=} \lim_{x\to+\infty} \frac{\cancel{2}\, e^{x^2}}{e^{x^2}\cdot \cancel{2}x} = 0$$

$\ulcorner$ зашто ако $f'(x) \to A > 0$, $x \to +\infty \Rightarrow f(x) \to +\infty$, $x \to +\infty$

$\forall \varepsilon > 0 \quad \exists M > 0 \quad x > M \quad f'(x) \in (A - \varepsilon, A + \varepsilon)$

$\varepsilon = \frac{A}{2} \qquad A - \frac{A}{2} = \frac{A}{2} > 0$

$\downarrow$

$M\left(\frac{A}{2}\right) = M$

$x > M \Rightarrow f(x) - f(M) = f'(\xi)\cdot(x - M) \quad , \xi \in (M, x) \quad , \xi > M \quad , f'(\xi) \in \left(\frac{A}{2}, \frac{3A}{2}\right)$

$$\Rightarrow f(x) - f(M) \geq \frac{A}{2}\underbrace{(x - M)}_{>0}$$

$$\underbrace{f(x)} \geq \underbrace{\frac{A}{2}(x - M)}_{>0} + \underbrace{f(M)}_{\in \mathbb{R}} \underset{x\to+\infty}{\longrightarrow} +\infty$$

$\Rightarrow f(x) \to +\infty . \lrcorner$