(1) $\displaystyle\int \frac{\arcsin x}{x^2}\,dx = \ulcorner\, u = \arcsin x \to du = \frac{dx}{\sqrt{1-x^2}}$

$\displaystyle\quad dv = \frac{dx}{x^2} \to v = -\frac{1}{x} \,\lrcorner$

$\displaystyle = -\frac{\arcsin x}{x} + \int \frac{dx}{x\sqrt{1-x^2}} = \ulcorner\, t = \arcsin x$

$\displaystyle\quad x = \sin t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\displaystyle\quad \sqrt{1-\sin^2 t} = \cos t \,\lrcorner$

$\displaystyle = -\frac{\arcsin x}{x} + \int \frac{\cos t\,dt}{\sin t\,\cos t}\cdot\frac{\sin t}{\sin t} = \ulcorner\, u = \cos t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\quad du = -\sin t\,dt$

$\quad \sin^2 t = 1 - \cos^2 t \,\lrcorner$

$\displaystyle = -\frac{\arcsin x}{x} + \int \frac{-du}{1-u^2} = -\frac{\arcsin x}{x} - \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C$

$\displaystyle\ulcorner\ \frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)$

$\displaystyle = -\frac{\arcsin x}{x} - \frac{1}{2}\ln\left|\frac{1+\overbrace{\cos(\arcsin x)}^{=\sqrt{1-x^2}}}{1-\underbrace{\cos(\arcsin x)}_{=\sqrt{1-x^2}}}\right| + C$

(2) $\displaystyle\int \frac{\ln(x+1) - \ln x}{x(x+1)}\,dx = \int \frac{\ln\frac{x+1}{x}}{\underbrace{\frac{x(x+1)}{x^2}}_{t}}\cdot\frac{dx}{x^2} = \ulcorner\, t = \frac{x+1}{x} = 1 + \frac{1}{x}$

$\displaystyle\quad dt = -\frac{1}{x^2}\,dx \,\lrcorner$

$\displaystyle = -\int \frac{\ln t}{t}\,dt = \ulcorner\, u = \ln t,\ du = \frac{dt}{t} \,\lrcorner = -\int u\,du = -\frac{u^2}{2} + C$

$\displaystyle = -\frac{\left(\ln\frac{x+1}{x}\right)^2}{2} + C$

(3) $\displaystyle\int \frac{\frac{e^x dx}{2}\cdot 2e^{-x}}{\sqrt{e^x+1}\,\left(e^x+1+1\right)} = \ulcorner\, t = \sqrt{e^x+1} \to t^2 = e^x + 1 \Rightarrow e^x = t^2 - 1$

$\displaystyle\quad dt = \frac{1}{2\sqrt{e^x+1}}\cdot e^x\,dx \,\lrcorner$

$\displaystyle = 2\int \frac{dt}{(t^2-1)(t^2+1)} = \ldots$

$*$: $\displaystyle I_n = \int \underbrace{\sin^n x\,dx}_{\underbrace{\sin^{n-1}x}_{u}\cdot\underbrace{\sin x\,dx}_{dv}} = \ulcorner\, u = \sin^{n-1}x \to du = (n-1)\sin^{n-2}x\cos x\,dx$

$\quad dv = \sin x\,dx \to v = -\cos x \,\lrcorner$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cdot \underbrace{\cos^2 x}_{1-\sin^2 x}\, dx$$

$$= -\cos x \cdot \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\boxed{I_n = \frac{1}{n-2}\left(-\cos x \cdot \sin^{n-1} x + (n-1) I_{n-2}\right)}$$

(4) Одредити $f:\mathbb{R}\to\mathbb{R}$ диф. тако да $f'(\ln x) = \begin{cases} 1, & 0<x\le 1 \\ x, & x\ge 1 \end{cases}$

$f(0)=0$.

$$f(t) \overset{\text{``}}{=} \int f'(t)\,dt = \left\{ \begin{matrix} t=\ln x \to x = e^t, \\ dt = \frac{dx}{x} \end{matrix} \right\} = \int f'(\ln x)\frac{dx}{x} =$$

до на константу

$$= \begin{cases} \int \frac{dx}{x}, & 0<x\le 1 \\ \int dx, & x\ge 1 \end{cases} = \begin{cases} \ln x + c_1, & 0<x\le 1 \quad (t\le 0) \\ x + c_2, & x\ge 1 \quad (t\ge 0) \end{cases}$$

$$= \begin{cases} t + c_1, & t\le 0 \\ e^t + c_2, & t\ge 0 \end{cases}$$

f непр $\Rightarrow$ $0 + c_1 = e^0 + c_2$

$c_1 = 1 + c_2$

$f(0)=0 \Rightarrow 0 + c_1 = 0 \Rightarrow c_1 = 0, \ c_2 = -1$.

$$f(t) = \begin{cases} t, & t\le 0 \\ e^t - 1, & t\ge 0 \end{cases}.$$

За вежбу:

$$\int \frac{dx}{(x+3)\sqrt{x^2+5x+8}}, \quad \int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx, \quad \int \frac{x^3\,dx}{\sqrt{1+2x-x^2}}, \quad \int\frac{dx}{\sqrt{1+e^{2x}}}$$

$$\int \frac{dx}{\sqrt[4]{1+x^4}}, \quad \int \frac{dx}{(x+1)\sqrt{x^2+x}}$$

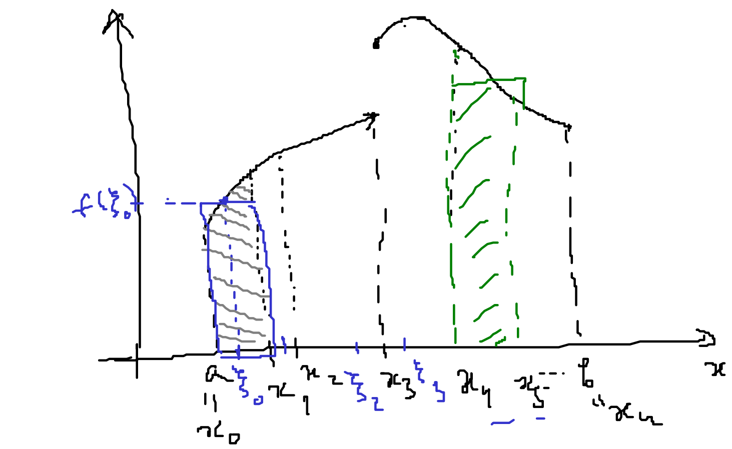
Одређени интеграли

$f : [a,b] \to \mathbb{R}$

$P: a = x_0 < x_1 < x_2 < \dots < x_n = b$

$\xi = \{\xi_k : \xi_k \in (x_k, x_{k+1}) : 0 \le k \le n-1\}$

$[x_k, x_{k+1}]$



$$\sigma(P, \xi, f) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot f(\xi_k)$$

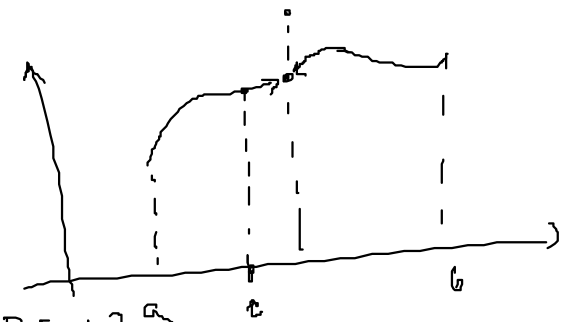
$$\lambda(P) = \max_{0 \le k \le n-1} (x_{k+1} - x_k)$$

$$\int_a^b f(x)\,dx = \lim_{\lambda(P) \to 0} \sigma(P, \xi, f)$$

→ ако постоји и коначан је онда је f Риман интеграбилна ($f \in \mathcal{R}[a,b]$) и $\int_a^b f(x)\,dx$ је Риманов интеграл фје f

- $C[a,b] \in \mathcal{R}[a,b]$
- f монотона $\Rightarrow f \in \mathcal{R}[a,b]$
- f има највише преб. много прекида $\Rightarrow f \in \mathcal{R}[a,b]$

  коначно



$f(x) = g(x)$ сем у највише преб. много тачака

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

* $f, g \in \mathcal{R}[a,b]$, α:

- $\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $c \in (a,b)$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $f(x) \cdot g(x) \in \mathcal{R}[a,b]$, $|f| \in \mathcal{R}[a,b]$, $|f| > \alpha > 0 \Rightarrow \frac{1}{f} \in \mathcal{R}[a,b]$

Њутн-Лајбницова ф-ла: $f \in C[a,b]$, $F(x) = \int f(x)\,dx$

$$\Rightarrow \int_a^b f(x)\,dx = F(b)-F(a) = F(x)\Big|_a^b .$$

парцијална интеграција: $u, v \in C^1[a,b]$

$$\int_a^b u(x)v'(x)\,dx = \left(u(x)v(x) - \int u'(x)v(x)\,dx \right)\Big|_a^b =$$

$$= u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx .$$

смена променљиве:

$$\int_a^b f(x)\,dx = \left[\begin{array}{l} x=\varphi(t) \to dx = \varphi'(t)\,dt \\ \varphi: [\alpha,\beta] \to [a,b] \\ \varphi \text{ бијекција}, \ \varphi \in C^1[\alpha,\beta] \\ \varphi(\alpha) = \underset{b}{a}, \ \varphi(\beta) = \underset{a}{b} \end{array}\right] = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\varphi'(t)\,dt$$

\- - - -

(1) $$\int_0^2 x^4\,dx = \frac{x^5}{5}\Big|_0^2 = \frac{32}{5} - 0 = \frac{32}{5}$$

(2) $$\int_1^{e^2} \underbrace{\frac{\ln x}{x}\,dx}_{dt} = \left[\begin{array}{l} t = \ln x \\ \text{бијекција на } [1,e^2] \\ \varphi(x) = \ln x \in C^1 \\ dt = \frac{dx}{x} \\ t(1) = \ln 1 = 0, \ t(e^2) = \ln e^2 = 2 \end{array}\right] = \int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = 2$$

(3) $$\int_1^3 \arcsin\sqrt{\frac{x}{x+1}}\,dx = \left[\begin{array}{l} u = \arcsin\sqrt{\frac{x}{x+1}} \to du = \frac{1}{\sqrt{1-\frac{x}{x+1}}}\cdot\frac{1}{2\sqrt{\frac{x}{x+1}}}\cdot\frac{dx}{(x+1)^2} \\ dv = dx \to v = x \end{array}\right]$$

$$\frac{1}{2}\cdot\frac{1}{\frac{\sqrt{x}}{(x+1)}}\cdot\frac{dx}{(x+1)^{\cancel{2}}}$$

$$= x\cdot\arcsin\sqrt{\frac{x}{x+1}}\Big|_1^3 - \frac{1}{2}\int_1^3 \frac{x\,dx}{\sqrt{x}\,(x+1)} = \left[\begin{array}{l} t = \sqrt{x} \\ dt = \frac{dx}{2\sqrt{x}} \\ \text{бијекција и } C^1[1,3] \\ t(1) = 1, \ t(3) = \sqrt{3} \end{array}\right]$$

$$= 3\arcsin\frac{\sqrt{3}}{2} - 1\cdot\arcsin\sqrt{\frac{1}{2}} - \int_1^{\sqrt{3}} \frac{t^2\,dt}{t^2+1} \quad (+1-1)$$

$$= 3\frac{\pi}{3} - \frac{\pi}{4} - \int_1^{\sqrt{3}} dt + \int_1^{\sqrt{3}} \frac{dt}{t^2+1} = \frac{3\pi}{4} - t\Big|_1^{\sqrt{3}} + \operatorname{arctg} t\Big|_1^{\sqrt{3}}$$

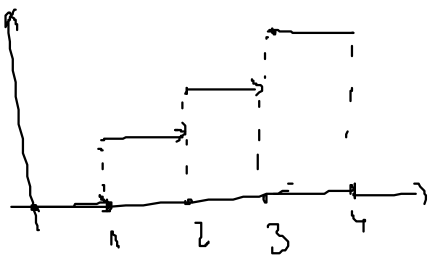
$$= \frac{3\pi}{4} - \sqrt{3} + 1 + \underbrace{\operatorname{arctg}\sqrt{3}}_{\pi/3} - \underbrace{\operatorname{arctg} 1}_{\pi/4} = \dots$$

(4) $$\int_{-3}^{0} |x^3+8|\, dx = \left[\begin{array}{ll} x^3+8 \geq 0 & x \geq -2 \\ x^3+8 \leq 0 & x \leq -2 \end{array}\right] = \int_{-3}^{-2} -(x^3+8)\, dx + \int_{-2}^{0} (x^3+8)\, dx$$

$$= -\left(\frac{x^4}{4} + 8x\right)\Big|_{-3}^{-2} + \left(\frac{x^4}{4} + 8x\right)\Big|_{-2}^{0} =$$

$$= -(4-16) + \frac{81}{4} - 24 + 0 - (4-16) = \frac{81}{4}$$

(5) $$\int_0^4 [x]\, dx = \int_0^1 \underbrace{[x]}_{0} dx + \int_1^2 \underbrace{[x]}_{1} dx + \int_2^3 \underbrace{[x]}_{2} dx + \int_3^4 \underbrace{[x]}_{3} dx$$



$$= 0 \cdot x\Big|_0^1 + 1 \cdot x\Big|_1^2 + 2 \cdot x\Big|_2^3 + 3x\Big|_3^4$$

$$= 6$$

(6) $a>0$ $$\int_0^a x^2\sqrt{a^2-x^2}\, dx = \left[\begin{array}{ll} x = a\sin t, & t \in [0, \frac{\pi}{2}] \\ \{a\cos t & t \in [0, \frac{\pi}{2}]\} \\ x(0) = 0 & [x(0) = a] \\ x(\frac{\pi}{2}) = a & [x(\frac{\pi}{2}) = 0] \\ \sqrt{a^2-x^2} = a\cos t \geq 0 & \end{array}\right] =$$

$$= \int_0^{\pi/2} a^2\sin^2 t\; a\cos t\; a\cos t\, dt = a^4 \int_0^{\pi/2} \underbrace{\sin^2 t\cos^2 t}_{\frac{1}{4}\sin^2 2t \,=\, \frac{1}{8}(1-\cos 4t)} dt$$

$$= \frac{a^4}{8}\left(t - \frac{1}{4}\sin 4t\right)\Big|_0^{\pi/2} = \frac{a^4}{8}\left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right) - \frac{a^4}{8}\left(0 - \frac{1}{4}\sin 0\right)$$

$$= \frac{a^4\pi}{16}$$