Ојлерове смене: $\int R(x, \sqrt{ax^2+bx+c})\,dx$

I: $\sqrt{ax^2+bx+c} = \pm\sqrt{a}\,x + t$

$a > 0$

$$\int \frac{\sqrt{4x^2+1} + 2x}{2x+3}\,dx = \left[\begin{aligned} &\sqrt{4x^2+1} = -2x + t \quad \uparrow^2 \\ &4x^2+1 = \cancel{4x^2} - 4xt + t^2 \\ &x = \frac{t^2-1}{4t} \\ &dx = \frac{1}{4}\,\frac{2t^2 - t^2 + 1}{t^2}\,dt = \frac{t^2+1}{4t^2}\,dt \end{aligned}\right]$$

$a = 4 > 0$

$$= \int \frac{-\cancel{2x} + t + \cancel{2x}}{2\frac{t^2-1}{4t} + 3}\cdot\frac{t^2+1}{4t^2}\,dt = \int \frac{t^2+1}{2t^2-2+12t}\,dt = \frac{1}{2}\int\frac{t^2+1}{t^2+6t-1}\,dt = \dots$$

II: $\sqrt{ax^2+bx+c} = \pm\sqrt{c} + xt \quad \uparrow^2$

$c > 0$

$$ax^2+bx+\cancel{c} = \cancel{c} \pm 2\sqrt{c}\,xt + x^2t^2$$

$$(b \mp 2\sqrt{c}\,t)\,x = x^2(t^2 - a)$$

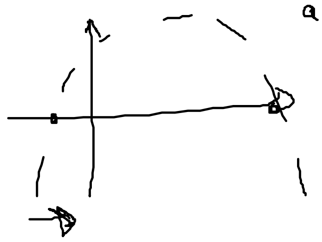
$$\Rightarrow \quad x = \frac{b \mp 2\sqrt{c}\,t}{t^2 - a}$$

2) $$\int \frac{dx}{1+\sqrt{1-2x-x^2}} = \left[\begin{aligned} &\sqrt{1-2x-x^2} = -1 + tx \\ &x = \frac{-2+2t}{t^2+1} \rightarrow dx = \frac{2t^2+2 - 2t(-2+2t)}{(t^2+1)^2}\,dt \end{aligned}\right]$$

$$= \int \frac{-2t^2+4t+2}{(t^2+1)^2\cdot t\,\frac{(-2+2t)}{\cancel{t^2+1}}}\,dt = \int \frac{-t^2+2t+1}{t(t-1)(t^2+1)}\,dt = \dots$$

III $a, c < 0$ $\quad \underbrace{\sqrt{ax^2+bx+c}}_{a(x-x_1)(x-x_2)} = t(x - x_1) \quad \uparrow^2$ $\qquad \left[\begin{aligned} &a(\cancel{x-x_1})(x-x_2) = t^2(x-x_1)^{\cancel{2}} \\ &x = \dots \end{aligned}\right]$

$x_1, x_2 \rightarrow$ нуле $ax^2+bx+c = 0$



3) $$\int \frac{dx}{2x + \sqrt{-x^2+5x-4}} = \left[\begin{aligned} &x^2 - 5x + 4 = 0 \\ &x_{1,2} = \frac{5 \pm \sqrt{25-16}}{2} \\ &x_1 = 1 \\ &x_2 = 4 \end{aligned}\right.$$

$$\sqrt{-(x^2-5x+4)} = t(x-1) \quad \uparrow^2$$

$$-(\cancel{x-1})(x-4) = t^2(x-1)^{\cancel{2}}$$

$$x = \frac{4+t^2}{t^2+1} = 1 + \frac{3}{t^2+1}$$

$$dx = \frac{3\cdot 2t}{(t^2+1)^2}\,dt \Big]$$

$$= \int \frac{\frac{6t}{(t^2+1)^2}\,dt}{\frac{8+2t^2}{t^2+1} + t\left(\frac{4+t^2}{t^2+1} - 1\right)} = \int \frac{6t\,dt}{8+2t^2+t(4+t^2-t^2-1)}$$

$$= \int \frac{6t\,dt}{2t^2+3t+8} = \dots$$

$$t = \sqrt{-\frac{x-4}{x-1}}$$

$$* \quad \int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}} \quad /'$$

$\deg P_n = n$

$\deg Q_{n-1} \leq n-1$

(4) $$\int \frac{2x^2+5x+1}{\sqrt{x^2+4x-3}}\,dx$$

$P_n(x) = 2x^2+5x+1 \qquad \deg P_n = 2$

$\deg Q_{n-1} \leq 1 \Rightarrow Q_{n-1}(x) = Ax+B$

$x \in \mathbb{R}$

$$\int \frac{2x^2+5x+1}{\sqrt{x^2+4x-3}}\,dx = (Ax+B)\sqrt{x^2+4x-3} + \lambda \int \frac{dx}{\sqrt{x^2+4x-3}} \quad /'$$

$$\frac{2x^2+5x+1}{\sqrt{x^2+4x-3}} = A\sqrt{x^2+4x-3} + (Ax+B)\frac{2x+4}{2\sqrt{x^2+4x-3}} + \frac{\lambda}{\sqrt{x^2+4x-3}}$$

$$= \frac{A(x^2+4x-3) + (Ax+B)(x+2) + \lambda}{\sqrt{x^2+4x-3}}$$

$x^2: \quad A+A = 2 \quad \to A = 1$

$x^1: \quad 4A+2A+B = 5 \quad \to B = -1$

$x^0: \quad -3A+2B+\lambda = 1 \quad \to \lambda = 6$

$$\Rightarrow \int \frac{2x^2+5x+1}{\sqrt{x^2+4x-3}}\,dx = (x-1)\sqrt{x^2+4x-3} + 6\underbrace{\int \frac{dx}{\sqrt{x^2+4x-3}}}_{I}$$

$u \geq 0, \quad u = \operatorname{arch}\frac{|t|}{\sqrt{7}}$

$$\int \frac{dx}{\sqrt{x^2+4x-3}} = \left[\begin{array}{l} x^2+4x-3 = (x+2)^2-7 \\ t = x+2 \\ dt = dx \end{array}\right] = \int \frac{dt}{\sqrt{t^2-7}} = \left[\begin{array}{l} t = \sqrt{7}\operatorname{ch}u, \; t > \sqrt{7}, \; dt = \sqrt{7}\operatorname{sh}u\,du \\ t = -\sqrt{7}\operatorname{ch}u, \; t < -\sqrt{7}, \; dt = -\sqrt{7}\operatorname{sh}u\,du \\ \sqrt{t^2-7} = \sqrt{7}\operatorname{sh}u > 0 \end{array}\right]$$

$$= \begin{cases} \int \frac{\sqrt{7}\operatorname{sh}u\,du}{\sqrt{7}\operatorname{sh}u} = \int du = u + C_1, & t > \sqrt{7} \\ -\int \frac{\sqrt{7}\operatorname{sh}u\,du}{\sqrt{7}\operatorname{sh}u} = -u + C_2, & t < -\sqrt{7} \end{cases} = \begin{cases} \operatorname{arch}\frac{t}{\sqrt{7}} + C_1 & t > \sqrt{7} \\ -\operatorname{arch}\frac{-t}{\sqrt{7}} + C_2 & t < -\sqrt{7} \end{cases}$$

$$\underset{t=x+2}{=} \dots$$

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx \underset{n\in\mathbb{N}}{=} \left\ulcorner t = \sqrt[n]{\frac{ax+b}{cx+d}} \quad \Big\uparrow^{n} \qquad t^n = \frac{ax+b}{cx+d} \right\lrcorner$$

① $$\int \frac{1-\overbrace{\sqrt{x+2}}^{=(x+2)^{1/2}}}{1+\underbrace{\sqrt[3]{x+2}}_{=(x+2)^{1/3}}}\, dx = \left\ulcorner \begin{array}{l} t^6 = x+2 \\ x = t^6-2 \\ dx = 6t^5\,dt \end{array} \right\lrcorner = \int \frac{1-t^3}{1+t^2}\cdot 6t^5\,dt = 6\int \frac{t^5-t^8}{1+t^2}\,dt$$

$$\frac{t^5-t^8}{1+t^2} = S(x) + \frac{R(x)}{1+t^2}, \quad \deg R < 2$$

$$t^8-t^5 : 1+t^2 = \underbrace{t^6-t^4-t^3+t^2+t-1}_{S(x)}$$

$$\begin{array}{l}
-\ t^8+t^6 \\
\hline
\quad -t^6-t^5 \\
-\ \ -t^6-t^4 \\
\hline
\qquad -t^5+t^4 \\
-\quad -t^5-t^3 \\
\hline
\qquad\quad t^4+t^3 \\
-\qquad t^4+t^2 \\
\hline
\qquad\qquad t^3-t^2 \\
-\qquad\quad t^3+t \\
\hline
\qquad\qquad\quad -t^2-t \\
-\qquad\qquad -t^2-1 \\
\hline
\qquad\qquad\qquad -t+1 = R(x)
\end{array}$$

$$\Rightarrow \int \frac{t^5-t^8}{1+t^2}\,dt = -\int (t^6-t^4-t^3+t^2+t-1)\,dt - \int \frac{-t+1}{t^2+1}\,dt$$

$$= \ldots$$

$$\downarrow$$

$$t = \sqrt[6]{x-2}$$

② $$\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}} \cdot \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\, dx = \int \frac{x+1+x-1-2\sqrt{x+1}\sqrt{x-1}}{\cancel{x}+1-(\cancel{x}-1)}\, dx$$

$$= \int \frac{2x-2\sqrt{x^2-1}}{2}\, dx = \int x\,dx - \int \sqrt{x^2-1}\,dx = \ldots$$

$$\downarrow \quad x = \frac{1}{\cos t}$$

$$\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}} \begin{array}{l} :\sqrt{x-1} \\ :\sqrt{x-1} \end{array} dx = \int \frac{\sqrt{\frac{x+1}{x-1}}-1}{\sqrt{\frac{x+1}{x-1}}+1}\, dx = \left\ulcorner t^2 = \frac{x+1}{x-1}, \ t = \sqrt{\frac{x+1}{x-1}} \right.$$

$$x = \frac{t^2-1}{t^2+1} = 1-\frac{2}{t^2+1} \to dx = \frac{4t}{(t^2+1)^2}\,dt$$

$$= \int \frac{t-1}{t+1} \cdot \frac{4t}{(t^2+1)^2}\, dt = \dots$$

③ $$\int \frac{dx}{\sqrt[3]{(x+1)(x+2)^2}} \qquad \sqrt[3]{(x+1)(x+2)^2 \cdot \frac{(x+2)}{(x+2)}} = \sqrt[3]{\frac{x+1}{x+2}} \cdot (x+2)$$

$$t = \sqrt[3]{\frac{x+1}{x+2}} \qquad t^3 = \frac{x+1}{x+2}$$

$$x = \frac{2t^3-1-1+1}{1-t^3} = -2 + \frac{1}{1-t^3}$$

$$dx = \frac{3t^2}{(1-t^3)^2}\, dt$$

$$= \int \frac{\frac{3t^2}{(1-t^3)^{\cancel{2}}}\, dt}{t \cdot \frac{1}{\cancel{(1-t^3)}}} = \int \frac{3t^2}{t(1-t^3)}\, dt = \dots$$

$\cdot\ I = \int x^p (a + bx^q)^r\, dx, \qquad p, q, r \in \mathbb{Q}\setminus\{0\}$

$1°\quad r \in \mathbb{Z} \qquad p = \frac{a}{b},\ q = \frac{c}{b} \quad \to \quad t = x^{1/b} \Rightarrow x = t^b$

$2°\quad r \notin \mathbb{Z} \qquad y = x^q \quad \to \quad dx = \frac{1}{q} y^{1/q - 1}\, dy$

$x = y^{1/q}$

$$I = \int y^{p/q} (a+by)^r \frac{1}{q} y^{1/q-1}\, dy = \frac{1}{q} \int y^{\frac{p+1}{q}-1} (a+by)^r\, dy$$

$2.1°\quad \frac{p+1}{q} \in \mathbb{Z},\quad r = \frac{i}{j},\ j \neq 1,\ i,j \in \mathbb{Z}$

$t = (a+by)^{1/j}, \quad a+by = t^j$

$y = \frac{t^j - a}{b}$

$2.2°\quad \frac{p+1}{q} + r \in \mathbb{Z}, \quad t = \left(\frac{a+by}{y}\right)^{1/j}, \quad y^{\frac{p+1}{q}-1+r} \cdot \frac{1}{y^r} \cdot (a+by)^r$

$r = \frac{i}{j}$

① $$\int \frac{\sqrt{x}}{1+\sqrt[5]{x}}\, dx = \int x^{1/2} (1 + x^{1/5})^{-1}\, dx = \left[ \begin{matrix} t = x^{1/10} \\ t^{10} = x \\ dx = 10t^9\, dt \end{matrix} \right] =$$

$$= \int t^5 (1+t^2)^{-1}\, 10t^9\, dt = 10 \int \frac{t^{14}}{1+t^2}\, dt = \underbrace{\dots}_{\text{za beżdy}}$$

② $$\int \sqrt[3]{3x - x^3}\, dx = \int (3x - x^3)^{1/3}\, dx = \int x^{1/3} (3 - x^2)^{1/3}\, dx = \left[ \begin{matrix} y = x^2 \\ x = \sqrt{y} \\ dx = \frac{1}{2} \frac{dy}{\sqrt{y}} \end{matrix} \right]$$

$r = 1/3 \notin \mathbb{Z}$

$$= \int y^{1/6} (3-y)^{1/3} \frac{1}{2} y^{-1/2} dy = \frac{1}{2} \int y^{-1/3} (3-y)^{1/3} dy = \frac{1}{2} \int y^{-1/3} \cdot y^{1/3} \left(\frac{3-y}{y}\right)^{1/3} dy$$

$$= \frac{1}{2} \int \left(\frac{3-y}{y}\right)^{1/3} dy = \left\lceil t^3 = \frac{3-y}{y} \quad y = \frac{3}{t^3+1} \quad dy = -\frac{3}{(t^3+1)^2} \cdot 3t^2\, dt \right\rfloor$$

$$= \frac{1}{2} \int t \cdot \frac{-9t^2}{(t^3+1)^2} dt$$

$$= -\frac{9}{2} \int \frac{t^3}{(t^3+1)^2} dt = \ldots$$

(3) $$\int \frac{dx}{(x+1)\sqrt{x^2+x}} \underset{x>0}{=} \int (x+1)^{-1} (\sqrt{x}\sqrt{x+1})^{-1} dx = \int x^{-1/2} (x+1)^{-3/2} dx = ⊛$$

$$\left\lceil t^2 = (x+1) \quad x = t^2 - 1 \quad dx = 2t\,dt \right\rfloor = \int (t^2-1)^{-1/2}\, t^{-3} \cdot 2t\,dt = 2\int \frac{dt}{t^2\sqrt{t^2-1}} = \ldots$$

II начин:

$$⊛ = \int x^{-1/2} \cdot x^{-3/2} \left(\frac{x+1}{x}\right)^{-3/2} dx = \int x^{-2} \left(\frac{x}{x+1}\right)^{3/2} dx$$

$$\left\lceil t^2 = \frac{x}{x+1} \quad x = \frac{-1+t^2+1}{1-t^2} = -1 + \frac{1}{1-t^2} \Rightarrow dx = \frac{2t}{(1-t^2)^2} dt \right\rfloor$$

$$= \underbrace{\int \frac{\cancel{(1-t^2)^2}}{\cancel{t^4}} \cdot \cancel{t^3} \frac{2\cancel{t}}{\cancel{(1-t^2)^2}} dt}_{2\int dt} = 2t + C = 2\sqrt{\frac{x}{x+1}} + C.$$