$\int R(x, \sqrt{ax^2+bx+c})\,dx \to \int R(t, \sqrt{\lambda^2-t^2})$

$\sqrt{\lambda^2+t^2}$, $\sqrt{\lambda^2-t^2}$, $\sqrt{t^2-\lambda^2}$

$\sqrt{x^2+x+1} = \sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}$

$y = x+\frac{1}{2}$, $\frac{3}{4} = \lambda$

$ch\,x = \dfrac{e^x+e^{-x}}{2}$, $sh\,x = \dfrac{e^x-e^{-x}}{2}$, $ch^2x - sh^2x = 1$

$\cdot \int R(x, \sqrt{\lambda^2+x^2})\,dx$

$\nearrow x = \lambda\,tg\,t$, $t\in(-\frac{\pi}{2},\frac{\pi}{2})$, $1+tg^2t = \dfrac{\cos^2t+\sin^2t}{\cos^2t} = \dfrac{1}{\cos^2t}$

$\searrow x = \lambda\,sh\,t$, $sh^2t+1 = ch^2t$.

$\lambda>0$

(1) $\int \sqrt{\lambda^2+x^2}\,dx = \left\lceil \begin{array}{l} x=\lambda\,tg\,t,\quad t\in(-\frac{\pi}{2},\frac{\pi}{2}) \quad t = arctg\,\frac{x}{\lambda} \\ dx = \dfrac{\lambda\,dt}{\cos^2t},\quad \sqrt{\lambda^2+\lambda^2tg^2t} = \lambda\cdot\dfrac{1}{\cos t} \end{array}\right\rfloor$

$= \int \dfrac{\lambda^2\,dt}{\cos^3t} = \underbrace{\ldots}_{\text{за вежбу}}$

(2) $\int \dfrac{dx}{\sqrt{1+x^2}} = \left\lceil \begin{array}{l} x = sh\,t,\quad t\in\mathbb{R} \to t = arsh\,x = \ln(x+\sqrt{x^2+1}) \\ dx = ch\,t\,dt,\quad \sqrt{1+x^2} = \sqrt{ch^2t} = |ch\,t| = ch\,t \end{array}\right\rfloor$

$= \int \dfrac{ch\,t\,dt}{ch\,t} = \int dt = t+C = arsh\,x + C = \ln(x+\sqrt{x^2+1})+C$

(3) $\int \dfrac{dx}{x\sqrt{1+x^2}} = \left\lceil \begin{array}{l} x = tg\,t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})\setminus\{0\} \\ dx = \dfrac{dt}{\cos^2t},\quad \sqrt{1+x^2} = \dfrac{1}{\cos t} \\ t = arctg\,x \end{array}\right\rfloor = \int \dfrac{\frac{dt}{\cos^2t}}{\frac{\sin t}{\cos t}\cdot\frac{1}{\cos t}} = \int\dfrac{dt}{\sin t} = \ldots$

$\cdot \int R(x, \sqrt{x^2-\lambda^2})\,dx$, $x\in\mathbb{R}\setminus(-\lambda,\lambda)$

$\nearrow x = \dfrac{\lambda}{\cos t}$, $t\in(0,\pi)\setminus\{\frac{\pi}{2}\}$, $\sqrt{x^2-\lambda^2} = \sqrt{\dfrac{\lambda^2\sin^2t}{\cos^2t}} = \dfrac{\lambda\sin t}{|\cos t|}$

$\searrow x\geq\lambda \to x = \lambda\,ch\,t$; $x\leq-\lambda \to x = -\lambda\,ch\,t$; $t\geq0$; $\sqrt{x^2-\lambda^2} = \lambda|sh\,t| = \lambda\,sh\,t$

$\frac{\lambda}{x} = \sin t \in(-1,1)$, $x\in(-\infty,+\infty)\setminus(-\lambda,\lambda)$

$\dfrac{e^x+e^{-x}}{2} \to$ парна $\quad ch\,t : [0,\infty)\to[1,+\infty)$

(4) $\int\dfrac{dx}{\sqrt{x^2-1}} = \left\lceil \begin{array}{l} x = \dfrac{1}{\cos t},\quad t\in(0,\pi)\setminus\{\frac{\pi}{2}\} \\ dx = \dfrac{dt}{\cos^2t}\sin t \\ \sqrt{x^2-1} = \dfrac{\sin t}{|\cos t|} \end{array}\right\rfloor = \int\dfrac{\frac{\cancel{\sin t}\,dt}{\cos^2t}}{\frac{\cancel{\sin t}}{|\cos t|}} = \int\dfrac{dt}{|\cos t|} = \begin{cases} \int\dfrac{dt}{\cos t} & t\in(0,\frac{\pi}{2}) \quad +C_1 \\ -\int\dfrac{dt}{\cos t} & t\in(\frac{\pi}{2},\pi) \quad +C_2 \end{cases}$

$x\in(-\infty,-\lambda)\cup(\lambda,+\infty)$

(5) $\int\sqrt{x^2-1}\,dx = \begin{cases} x\geq1 & \begin{array}{l} dx = sh\,t\,dt \\ x = ch\,t,\quad t\geq0 \\ t = arch\,x \end{array} \\ x\leq-1 & \begin{array}{l} x = -ch\,t,\quad t\geq0 \\ dx = -sh\,t\,dt \end{array} \end{cases} = \ldots = \begin{cases} \int sh^2t\,dt, & x\geq1 \\ -\int sh^2t\,dt & x\leq-1 \end{cases}$

(I) $\int sh^2 t\, dt = \int \frac{ch 2t - 1}{2} dt = \frac{1}{4} sh 2t - \frac{t}{2} + C = \frac{1}{4} sh(2 arch\, x) - \frac{arch x}{2} + C$

$\sin^2 t = \frac{1 - \cos 2t}{2}$

$sh^2 t$, $ch 2t = \underset{1 + sh^2 t}{ch^2 t} + sh^2 t$

$\boxed{sh^2 t = \frac{ch 2t - 1}{2}}$

$\int ch u\, du = sh u + C$

$\int sh u\, du = ch u + C$

(II) $sh\, t = \frac{e^t - e^{-t}}{2}$

$\int \left(\frac{e^t - e^{-t}}{2}\right)^2 dt = \dots$

* Ојлерове смене:

$\int R(x, \sqrt{ax^2 + bx + c})\, dx$

I: $a > 0 \Rightarrow \sqrt{ax^2 + bx + c} = \pm \sqrt{a}\, x + t$

$x = \frac{t^2 - c}{b \mp 2\sqrt{a}\, t}$

(1) $\int \frac{dx}{x + \underbrace{\sqrt{x^2 + x + 1}}}$ $\quad a = 1 > 0$

$= \Big\lceil \sqrt{x^2 + x + 1} = -1 \cdot x + t \quad \uparrow^2 \quad \rightarrow t = \sqrt{x^2 + x + 1} + x$

$\cancel{x^2} + x + 1 = \cancel{x^2} - 2xt + t^2$

$x = \frac{t^2 - 1}{1 + 2t}, \quad dx = \frac{2t(1 + 2t) - 2(t^2 - 1)}{(1 + 2t)^2} dt \Big\rfloor$

$= \int \frac{2t^2 + 2t + 2 \quad dt}{\left(\underbrace{\frac{t^2 - 1}{1 + 2t}}_{x} - \left(\underbrace{\frac{t^2 - 1}{1 + 2t}}_{-x}\right) + t\right)(1 + 2t)^2} = \int \frac{2(t^2 + t + 1)}{t(1 + 2t)^2} dt = \underbrace{- - - -}_{\text{за вежбу}}$

(2) $\int \frac{dx}{(1 + x)\sqrt{x^2 + x + 1}} = \Big\lceil \sqrt{x^2 + x + 1} = x + t \quad \uparrow^2 \quad \rightarrow t = \sqrt{x^2 + x + 1} - x$

$\cancel{x^2} + x + 1 = \cancel{x^2} + 2xt + t^2$

$x = \frac{t^2 - 1}{1 - 2t} \qquad dx = \frac{2t(1 - 2t) + 2(t^2 - 1)}{(1 - 2t)^2} dt \Big\rfloor$

$= \int \frac{\frac{-2t^2 + 2t - 2}{(1 - \cancel{2t})^{\cancel{2}}} dt}{\left(1 + \underbrace{\frac{t^2 - 1}{1 - \cancel{2t}}}_{x}\right)\left(\frac{t^2 - 1}{1 - \cancel{2t}} + t\right)} = \int \frac{\cancel{-}2(\cancel{t^2 - t + 1})\, dt}{(\cancel{1 - 2t + t^2 - 1})\underbrace{(\cancel{t^2} - 1 + t - \cancel{2}t^2)}_{(-1 + \cancel{t} - t^2)}} = \int \frac{2\, dt}{t(t - 2)}$

$= \dots$