(1) $\int \frac{\overbrace{\sin^3 x}^{=\sin^2 x\cdot \sin x}}{\cos x \sqrt[3]{\cos x}}dx = $ $\left[\begin{array}{l} t=\cos x \\ dt=-\sin x\,dx \\ \sin^2 x = 1-\cos^2 x = 1-t^2 \end{array}\right.$ $= -\int \frac{1-t^2}{t\cdot t^{1/3}}dt = -\int \frac{dt}{t^{4/3}} + \int \frac{t^2 dt}{t^{4/3}}$ ($\frac{t^2}{t^{4/3}} = t^{2/3}$)

$\frac{P(\sin x,\cos x)}{Q(\sin x,\cos x)}$, $P(\sin x,\cos x)=\sin^3 x$

$= -\frac{t^{-1/3}}{-1/3} + \frac{t^{5/3}}{5/3} + C = 3\frac{1}{\sqrt[3]{t}} + \frac{3}{5}\sqrt[3]{t^5} + C =$

$= \frac{3}{\sqrt[3]{\cos x}} + \frac{3}{5}\sqrt[3]{\cos^5 x} + C$

за пример за часа $\int \frac{dx}{\cos^3 x} = \left[t=\sin x \right.$ $\underbrace{\text{тј. } R \text{ непарна по } \sin x}$

- $\int \underbrace{\frac{P(\sin x,\cos x)}{Q(\sin x,\cos x)}}_{R(\sin x,\cos x)}dx$, P и Q различите парности по $\sin x$ $\Rightarrow t=\cos x$
P и Q —||— по $\cos x$ $\Rightarrow t=\sin x$.

(2) $\int \underbrace{tg^4 x}_{R(\sin x,\cos x)}dx = \int \frac{\sin^4 x}{\cos^4 x}dx = (*)$

$R(\sin x,\cos x) \to$ парна и по $\sin x$ и по $\cos x$ $\Rightarrow$ нећу претходне смене

$(*) = \left[\begin{array}{l} t=tg\, x \\ dt=\frac{dx}{\cos^2 x} \end{array}\right. = \int \underbrace{\frac{\sin^2 x}{\cos^2 x}}_{t^2}\cdot \underbrace{\sin^2 x}_{\frac{t^2}{1+t^2}} \underbrace{\frac{dx}{\cos^2 x}}_{dt} = \int \frac{t^4}{1+t^2}dt = \int \frac{t^2(t^2+1-1)}{1+t^2}dt$

$t^2 = \frac{\sin^2 x}{\cos^2 x} = \frac{\sin^2 x}{1-\sin^2 x}$
$\sin^2 x = \frac{t^2}{1+t^2}$

$= \int \left(t^2 - \frac{t^2+1-1}{1+t^2}\right)dt = \int \left(t^2 - 1 + \frac{1}{t^2+1}\right)dt =$

$= \frac{t^3}{3} - t + \operatorname{arctg} t + C = \frac{tg^3 x}{3} - tg\, x + \underbrace{\operatorname{arctg}(tg\, x)}_{\neq x,\ x\in\mathbb{R}} + C$

али ако би $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ онда $\operatorname{arctg}(tg\, x)=x$

$tg(\operatorname{arctg} x) = x$, $x\in\mathbb{R}$

(3) $\int \frac{dx}{\underbrace{\sin^4 x+\cos^4 x}_{\underbrace{(\sin^2 x+\cos^2 x)^2}_{1} - \underbrace{2\sin^2 x\cos^2 x}_{\frac{1}{2}\sin^2 2x}}} = \int \frac{dx}{1-\frac{1}{2}\sin^2 2x} = \left[\begin{array}{l} t=tg\, 2x \\ dt=\frac{dx}{\cos^2 2x}\cdot 2 \\ x=\frac{1}{2}\operatorname{arctg} t + C \\ \Rightarrow dx=\frac{1}{2}\frac{dt}{1+t^2} \\ \sin^2 2x = \frac{t^2}{1+t^2} \end{array}\right.$

$= \int \frac{\frac{1}{2}\frac{dt}{1+t^2}}{1-\frac{1}{2}\frac{t^2}{1+t^2}}\cdot\frac{2}{2} = \int \frac{dt}{2(1+t^2)-t^2} = \int \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{tg\, 2x}{\sqrt{2}} + C.$

(4) $\int \frac{dx}{1+a\cos x} = \left[tg\, x = \frac{\sin x}{\cos x} \right.$ $\to \cos^2 x = \frac{1}{1+t^2}$
$\cos x = \pm\frac{1}{\sqrt{1+t^2}}$ ← ова смена није најпогоднија

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} =$

$= \frac{1}{1+tg^2\frac{x}{2}} - \frac{tg^2 x/2}{1+tg^2 x/2} = \underline{\underline{\frac{1-tg^2 x/2}{1+tg^2 x/2}}}$

$t = tg\, x/2$
$dt = \frac{\frac{1}{2}dx}{\cos^2 x/2} \Rightarrow dx = 2\frac{dt}{1+t^2}$

$$= 2\int \frac{\frac{dt}{1+t^2}}{1+a\frac{1-t^2}{1+t^2}} = 2\int \frac{dt}{1+t^2+a-at^2} = 2\int \frac{dt}{(1+a)+(1-a)t^2} = I_a$$

1° $a=-1$

$$I_a = I_{-1} = 2\int \frac{dt}{2t^2} = -\frac{1}{t}+C = -\frac{1}{tg\,x/2}+C = -ctg\,x/2 + C$$

2° $a=1$

$$I_1 = 2\int \frac{dt}{2} = t + C = tg\,x/2 + C$$

3° $a<-1$

$$I_a = 2\int \frac{dt}{\underbrace{(1+a)}_{<0}+\underbrace{(1-a)}_{>0}t^2} = \frac{2}{(1-a)}\int \frac{dt}{t^2-\underbrace{\frac{1+a}{a-1}}_{>0,\ =\lambda^2}} = \frac{2}{(1-a)}\cdot\underbrace{\frac{1}{\frac{1+a}{a-1}}}_{=\frac{1}{\lambda^2}}\cdot\lambda\int\frac{\frac{1}{\lambda}dt}{\left(\frac{t}{\lambda}\right)^2-1} =$$

$u=\frac{t}{\lambda}$

$$= \frac{2}{1-a}\ \frac{a-1}{a+1}\ \frac{1}{2}\ln\left|\frac{1-\frac{t}{\lambda}}{1+\frac{t}{\lambda}}\right| + C \overset{\lambda>0}{=} \dots$$

$\lambda = \sqrt{\frac{1+a}{1-a}}$, $t = tg\,x/2$

4° $-1<a<1$

$$I_a = 2\int\frac{dt}{\underbrace{(1+a)}_{>0}+\underbrace{(1-a)}_{>0}t^2} = \frac{2}{1-a}\int\frac{dt}{\underbrace{\frac{1+a}{1-a}}_{>0}+t^2} = \frac{2}{1-a}\cdot\frac{1}{\sqrt{\frac{1+a}{1-a}}}\ arctg\frac{t}{\sqrt{\frac{1+a}{1-a}}} + C \overset{*}{=} \dots$$

$t = tg\,x/2$

5° $a>1$

$$I_a = 2\int\frac{dt}{\underbrace{(1+a)}_{>0}+\underbrace{(1-a)}_{<0}t^2} = -2\int\frac{dt}{(a-1)t^2-(1+a)} = -\frac{2}{1-a}\cdot\frac{a-1}{a+1}\cdot\frac{1}{2}\ln\left|\frac{1-\frac{t}{\lambda}}{1+\frac{t}{\lambda}}\right| + C$$

$= \dots$

(5) $$\int\frac{dx}{1+\sin x-\cos x} =$$

$\ulcorner$ $t = tg\,x/2 \rightarrow dx = 2\frac{dt}{1+t^2}$

$\sin x = 2\sin x/2\cos x/2 = 2\underbrace{\frac{\sin x/2}{\cos x/2}}_{tg\,x/2}\cdot\cos^2 x/2 = \frac{2t}{1+t^2}$

$\cos^2 x/2 = \frac{1}{1+tg^2 x/2}$

$\cos x = \frac{1-tg^2 x/2}{1+tg^2 x/2} = \frac{1-t^2}{1+t^2}$ $\lrcorner$

$$= 2\int\frac{\frac{dt}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} = 2\int\frac{dt}{1+t^2+2t-1+t^2} = 2\int\frac{dt}{2t^2+2t} = \int\frac{dt}{t(t+1)} =$$

$$= \int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt = \ln|t|-\ln|t+1|+C \overset{*}{=} \dots$$

$t = tg\,x/2$

(6) $$\int\frac{\sqrt{tg\,x}}{\sin x\cos x}dx =$$

$\ulcorner$ I начин смена $t = tg\,x/2$ ... $\lrcorner$

$\ulcorner$ $tg\,x>0 \rightarrow \frac{\sin x}{\cos x}>0 \Rightarrow \sin x\cdot\cos x>0$

$\sin^2 x = \frac{tg^2 x}{1+tg^2 x}$, $\cos^2 x = \frac{1}{1+tg^2 x} \rightarrow \sin x = \frac{tg\,x}{\sqrt{1+tg^2 x}}$, $\cos x = \pm\frac{1}{\sqrt{1+tg^2 x}}$

$\hookrightarrow \sin x \cdot \cos x = \dfrac{\operatorname{tg} x}{\sqrt{1+\operatorname{tg}^2 x}\sqrt{1+\operatorname{tg}^2 x}} = \dfrac{\operatorname{tg} x}{1+\operatorname{tg}^2 x}$

$t = \operatorname{tg} x \to x = \operatorname{arctg} t + C$

$dx = \dfrac{dt}{1+t^2}, \quad t>0$

$$= \int \frac{\sqrt{t}\,\frac{dt}{1+t^2}}{\frac{t}{1+t^2}} = \int \frac{dt}{\sqrt{t}} = \frac{t^{1/2}}{1/2} + C = 2\sqrt{t} + C = 2\sqrt{\operatorname{tg} x} + C$$

- за вежбу $\displaystyle\int \frac{dx}{\sin^4 x \cos^2 x}$.

* Интеграција ирационалних ф-ја

- $\int R(x, \sqrt{ax^2+bx+c})\,dx$, R – рационална ф-ја

$\sqrt{ax^2+bx+c} = \ulcorner t = \text{смена}, \; t = x+\frac{b}{2} \lrcorner = \sqrt{at^2+d} \; \ulcorner u = |a|t \lrcorner$

$\xrightarrow{d>0,\, a<0} \sqrt{\lambda^2 - u^2}, \quad u \in [-\lambda, \lambda]$

$\xrightarrow{d>0,\, a>0} \sqrt{\lambda^2+u^2}, \quad u \in \mathbb{R}$

$\xrightarrow{d<0,\, a>0} \sqrt{u^2-\lambda^2}, \quad u \in \mathbb{R}\setminus(-\lambda,\lambda)$

$\boxed{\text{I}}$ $\int R(x, \sqrt{\lambda^2-x^2})\,dx$, $x\in(-\lambda,\lambda)$, $x = \lambda \sin t$, $t\in(-\frac{\pi}{2},\frac{\pi}{2})$, $\cos t>0$

$\sqrt{\lambda^2-x^2} = \sqrt{\lambda^2-\lambda^2\sin^2 t} = \sqrt{\lambda^2\cos^2 t} = \lambda|\cos t| = \lambda\cos t$

$= \int R(\sin t, \lambda\cos t)\,\lambda\cos t\,dt \qquad dx = \lambda\cos t\,dt$

① $\displaystyle\int \sqrt{\lambda^2-x^2}\,dx = \ulcorner x = |\lambda|\sin t,\; t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\; t = \arcsin\frac{x}{|\lambda|} \lrcorner = \int \underbrace{|\lambda|\cos t}_{\sqrt{\lambda^2-x^2}} \cdot \underbrace{|\lambda|\cos t\,dt}_{dx} = \lambda^2 \int \cos^2 t\,dt$

$\displaystyle = \lambda^2 \int \frac{1+\cos 2t}{2}\,dt = \lambda^2 \frac{t}{2} + \frac{1}{4}\lambda^2 \sin 2t + C =$

$\displaystyle = \frac{\lambda^2 \arcsin\frac{x}{|\lambda|}}{2} + \frac{1}{4}\lambda^2 \sin 2\left(\arcsin\frac{x}{|\lambda|}\right) + C.$

② $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}} = \ulcorner \lambda = 2,\; x = 2\sin t,\; t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to t = \arcsin\frac{x}{2}$

$\sqrt{4-x^2} = 2\cos t$

$x^2 = 4\sin^2 t$

$dx = 2\cos t\,dt \lrcorner$

$\displaystyle = \int \frac{\cancel{2\cos t}\,dt}{4\sin^2 t \cdot \cancel{2\cos t}} = \frac{1}{4}\int\frac{dt}{\sin^2 t} = -\frac{1}{4}\operatorname{ctg} t + C$

$\displaystyle = -\frac{1}{4}\operatorname{ctg}\left(\arcsin\frac{x}{2}\right) + C.$

③ $\displaystyle\int \frac{3x+2}{\sqrt{3-4x-x^2}}\,dx = \ulcorner 3-4x-x^2 = 7-(x+2)^2$

$-(x^2+4x+4)+4$

$u = x+2, \quad x = u-2$

$du = dx \lrcorner$ $\displaystyle = \int \frac{3u-4}{\sqrt{7-u^2}}\,du$

$= \ulcorner u = \sqrt{7}\sin t,\; t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$t = \arcsin\frac{u}{\sqrt{7}}$,

$du = \sqrt{7}\cos t\,dt \lrcorner$ $\displaystyle = \int \frac{3\sqrt{7}\sin t - 4}{\sqrt{7}\cancel{\cos t}}\sqrt{7}\cancel{\cos t}\,dt$

$$= \int (3\sqrt{7}\sin t - 4)\,dt = 3\sqrt{7}\cos t - 4t + c = \dots$$

* Хиперболичке ф-је

$\sin x$, $\cos x$:

$\sin^2 x + \cos^2 x = 1$

$(\sin x)' = \cos x$ $\quad$ $\sin(x+y) = \sin x\cos y + \cos x \sin y$

$(\cos x)' = -\sin x$ $\quad$ $\cos(x+y) = \cos x\cos y - \sin x \sin y$

$\arcsin x, \arccos x \rightarrow$ инверзне $\sin x$ и $\cos x$

$$(e^x)' = e^x$$

$$\left.\begin{aligned} \operatorname{sh} x &= \frac{e^x - e^{-x}}{2} \\ \operatorname{ch} x &= \frac{e^x + e^{-x}}{2} \end{aligned}\right\} \quad \begin{aligned} (\operatorname{sh} x)' &= \operatorname{ch} x \\ (\operatorname{ch} x)' &= \operatorname{sh} x \end{aligned}$$

$$\left.\begin{aligned} \operatorname{sh}^2 x &= \frac{e^{2x} + e^{-2x} - 2}{4} \\ \operatorname{ch}^2 x &= \frac{e^{2x} + e^{-2x} + 2}{4} \end{aligned}\right\} \Rightarrow \operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$$

$$\left.\begin{aligned} \operatorname{sh}(x+y) &= \operatorname{sh} x \operatorname{ch} y + \operatorname{ch} x \operatorname{sh} y \\ \operatorname{ch}(x+y) &= \operatorname{ch} x \operatorname{ch} y + \operatorname{sh} x \operatorname{sh} y \end{aligned}\right\} \text{ проверити за венжбу}$$

$\operatorname{sh}: \mathbb{R} \rightarrow \mathbb{R}$

$$y = \operatorname{sh} x = \frac{e^x - e^{-x}}{2} = \frac{e^{-x}}{2} \cdot (e^{2x} - 1)$$

$$2e^x \cdot y = e^{2x} - 1 \rightarrow e^{2x} - 2e^x \cdot y - 1 = 0$$

$$t = e^x > 0 \rightarrow t^2 - 2y \cdot t - 1 \Rightarrow t_{1,2} = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2+1}$$

$$\Rightarrow t = y + \sqrt{y^2+1} = e^x$$

$$y = \operatorname{sh} x \quad \Rightarrow x = \ln(y + \sqrt{y^2+1}) = \operatorname{arsh} x$$

$\operatorname{arsh}: \mathbb{R} \rightarrow \mathbb{R}$

$$\int \frac{dt}{\sqrt{1+t^2}} = \ln|t + \sqrt{t^2+1}| + c = \operatorname{arsh} t$$

$\operatorname{ch}: \mathbb{R} \rightarrow [1, +\infty)$

$$y = \operatorname{ch} x = \frac{e^x + e^{-x}}{2} \underset{AГ}{\geq} \sqrt{e^x \cdot e^{-x}} = 1$$

ch није 1-1 на $\mathbb{R}$ $\quad$ $y = \operatorname{ch} x$

$\operatorname{ch} x = \operatorname{ch}(-x) \rightarrow$ парна $\quad$ $x = \operatorname{arch} y$

$\operatorname{ch}: [0, +\infty) \rightarrow [1, +\infty) \rightarrow$ бијекција, $\operatorname{arch}: [1, +\infty) \rightarrow [0, +\infty)$

$\operatorname{sh}(-x) = -\operatorname{sh} x \rightarrow$ непарна $\quad$ $\operatorname{arch} y = \ln(y + \sqrt{y^2-1})$

$$\int \frac{dt}{\sqrt{t^2-1}} = \ln|t + \sqrt{t^2-1}| + c = \operatorname{arch} t + c.$$