Метод Остроградског

$\int \frac{P(x)}{Q(x)}\,dx$, $\deg P < \deg Q$

$Q_1(x) = \text{нзд}\,(Q(x), Q'(x))$, $Q_2(x) = \frac{Q(x)}{Q_1(x)}$

$$\Rightarrow \int \frac{P(x)}{Q(x)}dx = \frac{X(x)}{Q_1(x)} + \underbrace{\int \frac{Y(x)}{Q_2(x)}dx}, \quad \begin{array}{l}\deg X < \deg Q_1\\ \deg Y < \deg Q_2\end{array}$$

1) $I = \int \frac{\overbrace{x^7+x^6+2x^4+2x^3+x+2}^{P(x)}}{\underbrace{(x^3+1)^2}_{Q}}dx = \int (x+1)\,dx + \underbrace{\int \frac{dx}{(x^3+1)^2}}$

$\deg P = 7 > \deg Q = 6$, $P(x) = S(x)\cdot Q(x) + R(x)$, $\deg R < \deg Q$

$$x^7+x^6+2x^4+2x^3+x+2 : x^6+2x^3+1 = \underbrace{x+1}_{S(x)}$$

$$\begin{array}{l} x^7+x^6+2x^4+2x^3+x+2\\ -\ (x^7+2x^4+x)\\ \hline \quad x^6+2x^3+2\\ \quad -\ (x^6+2x^3+1)\\ \hline \qquad 1 = R(x)\end{array}$$

$Q'(x) = 2(x^3+1)\cdot 3x^2$, $\text{нзд}\,(Q, Q') = x^3+1 = Q_1(x)$

$Q(x) = (x^3+1)^2$, $Q_2(x) = \frac{Q(x)}{Q_1(x)} = \frac{(x^3+1)^2}{x^3+1} = x^3+1$

$$I_1 = \int \frac{dx}{(x^3+1)^2} = \frac{X(x)}{\underbrace{x^3+1}_{Q_1(x)}} + \int \frac{Y(x)\,dx}{\underbrace{x^3+1}_{Q_2(x)}}$$

$\deg X, \deg Y \leq 2$

$X(x) = ax^2+bx+c$

$Y(x) = dx^2+ex+f$

$a, b, c, d, e, f \in \mathbb{R}$?

$$= \frac{ax^2+bx+c}{x^3+1} + \int \frac{dx^2+ex+f}{x^3+1}dx$$

$\rightarrow$ обо диференцирамо (тражимо извод)

$$\frac{0\cdot x^5+0\cdot x^4+0\cdot x^2+0\cdot x^3+0\cdot x+1\cdot x^0}{(x^3+1)^2} = \frac{(2ax+b)(x^3+1) - (ax^2+bx+c)\cdot 3x^2}{(x^3+1)^2} + \frac{dx^2+ex+f}{x^3+1} =$$

$$= \frac{(2ax^4+2ax+bx^3+b-3ax^4-3bx^3-3cx^2) + (dx^2+ex+f)(x^3+1)}{(x^3+1)^2}$$

x^5: $\boxed{d = 0}$

x^4: $2a - 3a + e = 0$ $\quad -a+e=0$

x^2: $-3c + d = 0 \Rightarrow \boxed{c=0}$

x^3: $b - 3b + f = 0$

x^1: $2a + e = 0 \Rightarrow a=0, e=0$

x^0: $b + f = 1$

$$\left.\begin{aligned} -2b+f&=0\\ b+f&=1\end{aligned}\right\} \Rightarrow 3b=1, \quad b=\tfrac{1}{3},\ f=\tfrac{2}{3} \quad \to X(x)=\tfrac{1}{3}x,\ Y(x)=\tfrac{2}{3}$$

$$I_1(x)=\frac{1}{3}\,\frac{x}{x^3+1}+\frac{2}{3}\underbrace{\int\frac{dx}{x^3+1}}_{I_2}$$

$$Q_2(x)=x^3+1=(x+1)\underbrace{(x^2-x+1)}_{\text{ово нема реалних нула}}$$

$$\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}=\frac{A(x^2-x+1)+(Bx+C)(x+1)}{x^3+1}$$

$$\begin{aligned}
x^2 &: \quad A+B=0 \quad \to B=-A\\
x &: \quad -A+B+C=0 \quad \to -2A+C=0\\
x^0 &: \quad A+C=1
\end{aligned}
\quad\Big\}\Rightarrow A=\frac{1}{3},\quad C=\tfrac{2}{3},\quad B=-\tfrac{1}{3}$$

$$I_2=\frac{1}{3}\underbrace{\int\frac{dx}{x+1}}_{\ln|x+1|}-\frac{1}{3}\underbrace{\int\frac{x-\overset{-\frac{1}{2}-\frac{3}{2}}{2}}{x^2-x+1}\,dx}_{I_3}$$

$x-\frac{1}{2}$

$$I_3=\int\frac{\frac{1}{2}(2x-1)}{x^2-x+1}\,dx-\frac{3}{2}\int\frac{dx}{x^2-x+1}$$

$$\frac{1}{2}\int\frac{2x-1}{x^2-x+1}\,dx=\left[\begin{aligned}t&=x^2-x+1\\ dt&=(2x-1)\,dx\end{aligned}\right]=\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\ln|t|\ {+C_1}=\frac{1}{2}\ln(x^2-x+1)+C_1$$

пре 2. часа

$$\frac{3}{2}\int\frac{dx}{\underbrace{x^2-x+1}_{t^2+a^2}}=\frac{3}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\underset{a^2}{\frac{3}{4}}}=\left[t=x-\tfrac{1}{2}\right]=\frac{3}{2}\int\frac{dt}{t^2+\frac{3}{4}}\overset{(3)}{\underset{a=\frac{\sqrt3}{2}}{=}}\frac{3}{2}\cdot\frac{1}{\frac{\sqrt3}{2}}\operatorname{arctg}\frac{t}{\frac{\sqrt3}{2}}+C$$

$t=t(x)$

$$=\sqrt3\operatorname{arctg}\frac{2x-1}{\sqrt3}+C_2$$

$$I=\frac{x^2}{2}+x+\frac{1}{3}\,\frac{x}{x^3+1}+\frac{2}{3}\left(\frac{1}{3}\ln|x+1|-\frac{1}{3}\left(\frac{1}{2}\ln(x^2-x+1)-\sqrt3\operatorname{arctg}\frac{2x-1}{\sqrt3}\right)\right)+C$$

(2) $\displaystyle\int\frac{dx}{x^4+1}$, $\quad Q(x)=x^4+1$

$Q'(x)=4x^3$ $\quad$ нзд$(Q,Q')=1$ $\Rightarrow$ не треба радити метод Остроградског јер немамо вишеструке нуле

$$Q(x)=x^4+1=(x^2+ax+b)(x^2+cx+d)$$

$x^3: \quad a + c = 0 \longrightarrow a = -c$

$x^2: \quad b + d + ac = 0$

$x^1: \quad bc + ad = 0 \longrightarrow a(d-b) = 0 \Rightarrow \underbrace{a = 0} \vee d - b = 0$

$x^0: \quad bd = 1$

$1^\circ \quad a = 0 \Rightarrow c = 0$

$\Rightarrow b + d = 0 \quad b = -d$

$\rightarrow -b^2 = 1 \quad \lightning \quad b \in \mathbb{R}$

$\Rightarrow a \neq 0 \Rightarrow d - b = 0$

$d = b \Rightarrow d^2 = 1 \rightarrow d = 1 \vee d = -1$

$2^\circ \quad d = 1:$

$1 + 1 + ac = 0$

$\left.\begin{array}{l} ac = -2 \\ a = -c \end{array}\right\} \Rightarrow -a^2 = -2 \Rightarrow \boxed{a = \pm\sqrt{2}}$

$3^\circ \quad d = -1:$

$\left.\begin{array}{l} ac = 2 \\ -a^2 = 2 \end{array}\right. \quad \lightning \quad a \in \mathbb{R}$

$$\Rightarrow x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

$$\int \frac{dx}{x^4+1} = \underbrace{\int \frac{Ax+B}{x^2+\sqrt{2}x+1}\,dx}_{I_1} + \underbrace{\int \frac{Cx+D}{x^2-\sqrt{2}x+1}\,dx}_{I_2} \Bigg/ \Rightarrow A, B, C, D = ?$$

$$\frac{1}{x^4+1} = \frac{(Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1)}{x^4+1}$$

$x^3: \quad A + C = 0$

$x^2: \quad -\sqrt{2}A + B + \sqrt{2}\cdot C + D = 0$

$\Downarrow$

$\sqrt{2}(C - A) = -1$

$A - C = \frac{1}{\sqrt{2}}$

$A = \frac{1}{2\sqrt{2}}, \quad C = -\frac{1}{2\sqrt{2}}$

$x: \quad A - \sqrt{2}B + C + \sqrt{2}D = 0 \Rightarrow D - B = 0$

$x^0: \quad B + D = 1$

$B = D = \frac{1}{2}$

$$I_1 = \int \frac{\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2+\sqrt{2}x+1}\,dx = \frac{1}{2\sqrt{2}} \int \frac{\frac{1}{2}\cdot 2x + \sqrt{2}}{\underbrace{x^2+\sqrt{2}x+1}_{t}}\,dx = \frac{1}{2\sqrt{2}} \int \frac{\frac{1}{2}(2x+\sqrt{2})}{x^2+\sqrt{2}x+1}\,dx + \frac{1}{2\sqrt{2}} \int \frac{\frac{\sqrt{2}}{2}\,dx}{x^2+\sqrt{2}x+1}$$

$t \rightarrow dt = 2x + \sqrt{2} \qquad t = x^2 + \sqrt{2}x + 1$

$$= \frac{1}{4\sqrt{2}} \int \frac{dt}{t} + \frac{1}{4} \int \frac{dx}{\underbrace{x^2+\sqrt{2}x+1}_{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}}, \quad a^2 = \frac{1}{2}$$

$$= \frac{1}{4\sqrt{2}} \ln|t| + \frac{1}{4} \cdot \frac{1}{\frac{1}{\sqrt{2}}} \operatorname{arctg} \frac{x + \frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} + C =$$

$$= \frac{1}{4\sqrt{2}} \ln(x^2+\sqrt{2}x+1) + \frac{\sqrt{2}}{4} \operatorname{arctg}(\sqrt{2}x + 1) + C$$

I_2 за вежбу.

(3) $$\int \frac{dx}{x^5+1}$$

$$\left.\begin{array}{l} Q(x) = x^5+1 \\ Q'(x) = 5x^4 \end{array}\right\} \Rightarrow$$ нема потребе радити Метод Остроградског

$$x^5+1 : x+1 = x^4 - x^3 + x^2 - x + 1$$

$$x^5+1 = (x+1)\underbrace{(x^4 - x^3 + x^2 - x + 1)}_{\text{(нема реалних нула)}}$$ → ово неће бити увек очигледно

$$x^4 - x^3 + x^2 - x + 1 = (x^2+ax+b)(x^2+cx+d)$$

$$\left.\begin{array}{ll} x^3: & a+c = -1 \\ x^2: & b+d+ac = 1 \\ x: & ad+bc = -1 \\ x^0: & bd = 1 \end{array}\right\}$$ није толико једноставно

$$x^4 - x^3 + x^2 - x + 1 = x^2\left(x^2 - x + 1 - \frac{1}{x} + \frac{1}{x^2}\right) =$$

$$t_{1/2} = \frac{1 \pm \sqrt{5}}{2}$$

$$= x^2\Big(\underbrace{x^2 + \frac{1}{x^2} + 2}_{t^2} - \underbrace{\left(x + \frac{1}{x}\right)}_{t} + 1 - 2\Big) = x^2(t^2 - t - 1)$$

$$= x^2\left(t - \frac{1+\sqrt{5}}{2}\right)\left(t - \frac{1-\sqrt{5}}{2}\right) = x^2\left(x + \frac{1}{x} - \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1}{x} - \frac{1-\sqrt{5}}{2}\right)$$

$$t = x + \frac{1}{x} \qquad x \cdot x$$

$$= \left(x^2 - \frac{1+\sqrt{5}}{2} x + 1\right)\left(x^2 - \frac{1-\sqrt{5}}{2} x + 1\right)$$

$$\frac{1}{x^5+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2 - \frac{1+\sqrt{5}}{2}x + 1} + \frac{Dx+E}{x^2 - \frac{1-\sqrt{5}}{2}x + 1} = \ldots$$

За вежбу завршити до краја ...

• $$\int R(\sin x, \cos x)\,dx, \quad R(u,s) = \frac{P(u,s)}{Q(u,s)}$$

$$\ulcorner t = \text{смена} \lrcorner = \underbrace{\int R_1(t)\,dt}_{\text{ово ћемо знати да израчунамо}}, \quad R_1(t) = \frac{P_1(t)}{Q_1(t)}, \quad P_1, Q_1 \text{ полиноми}$$

(1) $$\int \frac{dx}{\cos^3 x} = \int \frac{\cos x}{\underbrace{\cos^4 x}_{(1-\sin^2 x)^2}}\,dx = \begin{array}{l} \ulcorner t = \sin x \\ dt = \cos x\,dx \lrcorner \end{array} = \underbrace{\int \frac{dt}{(1-t^2)^2}}_{I} = (*)$$

$$\cos x = f(\underline{tg\, x}) = ?$$

$$\frac{\sin^2}{\cos^2} = tg^2 \rightarrow \frac{1-\cos^2}{\cos^2} = tg^2 \rightarrow \cos = \frac{1}{\sqrt{1-tg^2}}$$ → може и смена $tg\,x$, али и може и лакше

$$I = \int \frac{dt}{(1-t^2)^2} = \frac{X(t)}{1-t^2} + \int \frac{Y(t)}{1-t^2}\,dt$$

$Q(t) = (1-t^2)^2$, $Q'(t) = 2(1-t^2)\cdot(-2t)$

$Q_1(t) = \text{nzd}(Q, Q') = 1-t^2$, $Q_2(t) = 1-t^2 \Rightarrow \deg X, \deg Y \leq 1$

$X(t) = at+b$

$Y(t) = ct+d$

$$\frac{1}{(1-t^2)^2} = \frac{a(1-t^2)+2t(at+b)}{(1-t^2)^2} + \frac{ct+d}{(1-t^2)} = \frac{a+at^2+2bt+(ct+d)(1-t^2)}{(1-t^2)^2}$$

$t^3: \quad -c = 0 \qquad\qquad t: \quad 2b+c = 0 \Rightarrow b = 0$

$t^2: \quad a-d = 0 \qquad\qquad t^0: \quad a+d = 1 \Rightarrow a = d = \frac{1}{2}$

$$I = \frac{1}{2}\frac{t}{1-t^2} + \frac{1}{2}\int \frac{dt}{1-t^2} = \frac{1}{2}\frac{t}{1-t^2} + \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$