смена променљиве

$$\int f(g(x))\cdot g'(x)\,dx = \begin{bmatrix} t=g(x) \\ dt=g'(x)\,dx \end{bmatrix} = \int f(t)\,dt$$

парцијална интеграција

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)\,u'(x)\,dx$$

① $$\int x^2\sin(1-x)\,dx = \begin{bmatrix} u=x^2 \to du=u'(x)\,dx = 2x\,dx \\ dv=\sin(1-x)\,dx \\ v=\int \sin(1-x)\underbrace{dx}_{-dt} = \begin{bmatrix} t=1-x \\ dt=-dx \end{bmatrix} = -\int \sin t\,dt \\ = \cos t = \cos(1-x) \end{bmatrix} =$$

$$= x^2\cos(1-x) - 2\int x\cos(1-x)\,dx = \begin{bmatrix} u=x \Rightarrow du=dx \\ dv=\cos(1-x)\,dx \Rightarrow v=-\sin(1-x) \end{bmatrix} =$$

$$= x^2\cos(1-x) - 2\left(-x\sin(1-x) + \int \sin(1-x)\,dx\right) =$$

$$= x^2\cos(1-x) + 2x\sin(1-x) - 2\cos(1-x) + C.$$

$$\cdot \int p(x)\underset{\cos(\alpha x+\beta)}{\sin(\alpha x+\beta)}\,dx = \begin{bmatrix} u=p(x) \\ dv=\underset{\cos(\alpha x+\beta)}{\sin(\alpha x+\beta)}\,dx \end{bmatrix}$$

② $$\int x^n\ln x\,dx = \begin{bmatrix} u=\ln x \to du=\frac{1}{x}\,dx \\ dv=x^n\,dx \to v=\frac{x^{n+1}}{n+1} \end{bmatrix} = \frac{x^{n+1}}{n+1}\ln x - \int\frac{x^{n+1}}{n+1}\cdot\frac{1}{x}\,dx =$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + C$$

$$\cdot \int p(x)\ln q(x)\,dx = \begin{bmatrix} u=\ln q(x) \\ dv=p(x)\,dx \end{bmatrix}$$

③ $$\int e^{\sqrt{x}}\,dx = \begin{bmatrix} u=e^{\sqrt{x}} \to du=e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\,dx \\ dv=dx \to v=x \end{bmatrix} = x\cdot e^{\sqrt{x}} - \frac{1}{2}\int \sqrt{x}\,e^{\sqrt{x}}\,\frac{dx}{\sqrt{x}} \quad \begin{bmatrix} t=\sqrt{x} \\ dt=\frac{dx}{2\sqrt{x}} \end{bmatrix}$$

$$= x\cdot e^{\sqrt{x}} - \underbrace{\int t^2e^t\,dt}_{I_2} \overset{t=\sqrt{x}}{=} \cancel{x e^{\sqrt{x}}} - \cancel{x e^{\sqrt{x}}} + 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

$$I_2 = \int t^2e^t\,dt = \begin{bmatrix} u=t^2 \to du=2t\,dt \\ dv=e^t\,dt \to v=e^t \end{bmatrix} = t^2e^t - 2\int te^t\,dt = \begin{bmatrix} u=t \Rightarrow du=dt \\ dv=e^t\,dt \Rightarrow v=e^t \end{bmatrix}$$

$$= t^2e^t - 2te^t + 2\int e^t\,dt = t^2e^t - 2te^t + 2e^t + C$$

(4) $I_n=\int x^n e^x dx = \left[\begin{array}{l} u=x^n \rightarrow du = n x^{n-1}dx \\ dv=e^x dx \rightarrow v=e^x \end{array}\right] = x^n\cdot e^x - n\underbrace{\int x^{n-1}e^x dx}_{I_{n-1}}$

$I_1 = xe^x - e^x + c_1$

$I_2 = x^2e^x - 2I_1 = x^2e^x - 2xe^x + 2e^x + C_2$

$I_3 = x^3e^x - 3I_2 = x^3e^x - 3\underbrace{x^2e^x}\, + 2\cdot 3\, xe^x - 2\cdot 3 e^x + C_3$

...

$I_n = x^ne^x - nx^{n-1}e^x + n\cdot(n-1)x^{n-2}e^x - n\cdot(n-1)\cdot(n-2)x^{n-3}e^x + \dots$
$+ (-1)^{n-k}\, n\cdot(n-1)\cdots(n-k+1)\, x^{n-k}e^x + \dots + (-1)^n\, n!\, x^0 e^x + C_n$

$= e^x \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} x^{n-k} = e^x \sum_{i=0}^{n} (-1)^{n-i} \frac{n!}{i!} x^i, \quad n\in\mathbb{N}$

ПМИ ... - доказати да је ово заиста тачно!

(5) $I=\int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{3/2}}\, dx = \left[\begin{array}{l} t=\operatorname{arctg} x \Rightarrow x = \operatorname{tg} t \\ dt = \frac{dx}{1+x^2} \end{array}\right] =$

$t\in(-\frac{\pi}{2},\frac{\pi}{2}) \Rightarrow \cos t>0$

$(1+x^2)^{1/2}\cdot(1+x^2)$, dt

$= \int \frac{\operatorname{tg} t\cdot e^t}{(1+\operatorname{tg}^2 t)^{1/2}}\, dt = \int \frac{\operatorname{tg} t}{\left(\frac{1}{\cos^2 t}\right)^{1/2}} e^t\, dt = \int \frac{\operatorname{tg} t}{\frac{1}{\cos t}} e^t\, dt = \int \sin t\, e^t\, dt$

$\operatorname{tg} t = \frac{\sin t}{\cos t}$

$= \left[\begin{array}{l} u=\sin t \rightarrow du=\cos t\, dt \\ dv=e^t dt \rightarrow v=e^t \end{array}\right] = e^t\cdot\sin t - \int e^t \cos t\, dt =$

$= \left[\begin{array}{l} u=\cos t \rightarrow du=-\sin t\, dt \\ dv=e^t dt \rightarrow v=e^t \end{array}\right] = e^t\sin t - \cos t\, e^t - \underbrace{\int e^t \sin t\, dt}_{-I+c}$

$2I = e^t\sin t - \cos t\, e^t + c$

$I = \frac{e^t\sin t - e^t\cos t}{2} + C_1 \underset{\downarrow}{=} \frac{e^{\operatorname{arctg} x}\sin(\operatorname{arctg} x) - e^{\operatorname{arctg} x}\cos(\operatorname{arctg} x)}{2} + C_1$

$t=\operatorname{arctg}$

$t\in(-\frac{\pi}{2},\frac{\pi}{2}) \Rightarrow \cos t>0$

(6) $\int \arcsin^2 x\, dx = \left[\begin{array}{l} t=\arcsin x \rightarrow x=\sin t,\ dx=\cos t\, dt \\ dt = \frac{1}{\sqrt{1-x^2}}dx \\ dx = \sqrt{1-x^2}\,dt \\ = \sqrt{1-\sin^2 t}\, dt = \sqrt{\cos^2 t}\, dt \\ = |\cos t|\, dt = \cos t\, dt \end{array}\right] = \int t^2\cdot\cos t\, dt$

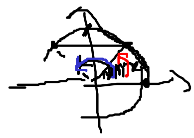
$x\in(-1,1)$

$= \left[\begin{array}{l} u=t^2 \rightarrow du=2t\,dt \\ dv=\cos t\, dt \rightarrow v=\sin t \end{array}\right] = t^2\sin t - 2\int t\sin t\, dt = \left[\begin{array}{l} u=t \rightarrow du=dt \\ dv=\sin t\, dt \rightarrow v=-\cos t \end{array}\right]$

$= t^2\sin t + 2\cos t\cdot t - 2\int \cos t\, dt = t^2\sin t + 2\cos t\cdot t - 2\sin t + c$

$$= (\arcsin x)^2 \underbrace{\sin(\arcsin x)}_{x} + 2\arcsin x \cdot \underbrace{\cos(\arcsin x)}_{\sqrt{1-x^2}} - 2x + C$$

$\pi - x$



$$\arcsin(\sin x) = \begin{cases} x - 2k\pi , & x \in \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right) \\ (\pi - x) - 2k\pi & x \in \left(-\frac{\pi}{2} + (2k+1)\pi, \frac{\pi}{2} + (2k+1)\pi\right) \end{cases}$$

$$\sin(\arcsin x) = x \quad \in (-1, 1)$$

$\cdot \displaystyle\int \frac{P(x)}{Q(x)}\,dx$

$\deg P \geq \deg Q \implies P(x) = S(x)\cdot Q(x) + R(x) \quad , \quad \deg R < \deg Q$

$$Q(x) = c\cdot(x-a_1)^{\alpha_1} \cdots (x-a_k)^{\alpha_k} \cdot (x^2 + c_1 x + b_1)^{\beta_1} \cdots (x^2 + c_l x + b_l)^{\beta_l}$$

$$(x - (a+ib))(x - (a-ib)) = (x^2 - 2ax + (a^2+b^2))$$

$$\alpha_1 + \dots + \alpha_k + 2\beta_1 + \dots + 2\beta_l = \deg Q$$

$$\frac{R(x)}{Q(x)} = \frac{A_1^1}{x-a_1} + \frac{A_1^2}{(x-a_1)^2} + \dots + \frac{A_1^{\alpha_1}}{(x-a_1)^{\alpha_1}} + \frac{A_2^1}{x-a_2} + \frac{A_2^2}{(x-a_2)^2} + \dots + \frac{A_2^{\alpha_2}}{(x-a_2)^{\alpha_2}} + \dots + \frac{A_k^{\alpha_k}}{(x-a_k)^{\alpha_k}} +$$
$$+ \frac{B_1^1 x + C_1^1}{(x^2+c_1x+b_1)} + \frac{B_1^2 x + C_1^2}{(x^2+c_1x+b_1)^2} + \dots + \frac{B_1^{\beta_1}x + C_1^{\beta_1}}{(x^2+c_1x+b_1)^{\beta_1}} + \dots + \frac{B_l^{\beta_l} x + C_l^{\beta_l}}{(x^2+c_lx+b_l)^{\beta_l}}$$

$$A_1^1, A_1^2, \dots, A_k^{\alpha_k}, B_1^1, C_1^1, \dots, B_l^{\beta_l}, C_l^{\beta_l} \in \mathbb{R}$$

(1) $\displaystyle\int \frac{3x^2+x+1}{(x^2-4x+4)(x^2-4x+5)}\,dx$

$P(x) = 3x^2 + x + 1 \qquad \deg P = 2 < \deg Q = 4$

$Q(x) = (x^2-4x+4)(x^2-4x+5)$

$Q(x) = (x-2)^2 (x^2-4x+5)^1 \implies \beta_1 = 1$

$$\frac{P(x)}{Q(x)} = \frac{A_1^1}{x-2} + \frac{A_1^2}{(x-2)^2} + \frac{B_1^1 x + C_1^1}{x^2-4x+5} = \frac{A_1^1(x-2)(x^2-4x+5) + A_1^2(x^2-4x+5) + (B_1^1x + C_1^1)\underbrace{(x-2)^2}_{(x^2-4x+4)}}{\underbrace{(x-2)^2(x^2-4x+5)}_{Q(x)}}$$

$$P(x) = \overset{0\cdot x^3 +}{} 3x^2 + x + 1 = x^3(A_1^1 + B_1^1) + x^2(-4A_1^1 - 2A_1^1 + A_1^2 - 4B_1^1 + C_1^1) + x(8A_1^1 + 5A_1^1 - 4A_1^2 + 4B_1^1 - 4C_1^1)$$
$$+ x^0(-10A_1^1 + 5A_1^2 + 4C_1^1)$$

$A_1^1 + B_1^1 = 0 \quad \to \quad A_1^1 = -B_1^1$

$-6A_1^1 + A_1^2 - 4B_1^1 + C_1^1 = 3 \quad \longrightarrow \quad -2A_1^1 + A_1^2 + C_1^1 = 3$

$13A_1^1 - 4A_1^2 + 4B_1^1 - 4C_1^1 = 1 \qquad 9A_1^1 - 4A_1^2 - 4C_1^1 = 1 \quad \cdot 4$

$-10A_1^1 + 5A_1^2 + 4C_1^1 = 1 \qquad -10A_1^1 + 5A_1^2 + 4C_1^1 = 1 \quad \cdot(-4)$

$-2A_1^1 + A_1^2 + C_1^1 = 3 \quad \Rightarrow C_1^1 = 3 - 15 + 26 = 14$

$\boxed{A_1^1 \qquad = 13} \qquad B_1^1 = -13$

$-2A_1^1 + A_1^2 \qquad = -11 \quad \Rightarrow A_1^2 = -11 + 2\cdot 13 = 15$

$$\int \frac{3x^2+x+1}{(x-2)^2(x^2-4x+5)}\,dx = \underbrace{13\int \frac{dx}{x-2}}_{13\ln|x-2|+C_1} + \underbrace{15\int \frac{dx}{(x-2)^2}}_{15\frac{(x-2)^{-1}}{-1}+C_2} + \underbrace{\int \frac{-13x+14}{x^2-4x+5}\,dx}_{I_3}$$

$$I_3 = \int \frac{-13x+14}{(x-2)^2+1}\,dx = \begin{bmatrix} t = x-2 \to x = t+2 \\ dt = dx \end{bmatrix} = \int \frac{-13t-26+14}{t^2+1}\,dt = \int \frac{-13t-12}{t^2+1}\,dt$$

$$= -13\int \frac{t\,dt}{t^2+1} - 12\int \frac{dt}{t^2+1} = -\frac{13}{2}\ln(t^2+1) - 12\operatorname{arctg} t + C_3$$

$$= -\frac{13}{2}\ln(x^2-4x+5) - 12\operatorname{arctg}(x-2) + C_3$$

$$I = 13\ln|x-2| - \frac{15}{x-2} - \frac{13}{2}\ln(x^2-4x+5) - 12\operatorname{arctg}(x-2) + C$$

2 $\displaystyle\int \frac{2x^2+2x+13}{(x-2)(x^2+1)^2}\,dx$

$P(x) = 2x^2+2x+13 \to \deg P = 2$

$Q(x) = (x-2)(x^2+1)^2 \to \deg Q = 5$

$$\frac{2x^2+2x+13}{Q(x)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} = \frac{A(x^4+2x^2+1) + (Bx+C)\overbrace{(x-2)(x^2+1)}^{(x^3-2x^2+x-2)} + (Dx+E)(x-2)}{(x-2)(x^2+1)^2}$$

$x^4: \quad A + B = 0 \to B = -A \qquad x: \quad -2B + C - 2D + E = 2$

$x^3: \quad -2B + C = 0 \to C = -2A \qquad x^0: \quad A \quad -2C \quad -2E = 13$

$x^2: \quad 2A + B - 2C + D = 2$

$5A + D = 2$

$-2D + E = 2$

$5A - 2E = 13 \quad \cdot(-1)$

$2E + D = -11$

$E - 2D = 2 \quad \cdot 2$

$5E = -20 \Rightarrow E = -4$

$D = -11 + 8 = -3$

$5A = 5 \Rightarrow A = 1$

$B = -1$

$C = -2$

$$\Rightarrow I = \int \frac{dx}{x-2} + \int \frac{-x-2}{x^2+1}\,dx + \underbrace{\int \frac{-3x-4}{(x^2+1)^2}\,dx}_{I_3 + I_4}$$

$\int \frac{dx}{x-2} = \ln|x-2|$

$\int \frac{-x-2}{x^2+1}\,dx = -\int \frac{x\,dx}{x^2+1} - 2\int \frac{dx}{x^2+1}$, with $\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln(x^2+1)$ and $-2\int \frac{dx}{x^2+1} = -2\operatorname{arctg} x$

$$I_3 = -3\int \frac{x\,dx}{(x^2+1)^2} = \left\lceil \begin{matrix} t = x^2+1 \\ dt = 2x\,dx \end{matrix} \right\rfloor = -\frac{3}{2}\int \frac{dt}{t^2} = \dots$$

$$I_4 = \underbrace{-4}\underbrace{\int \frac{dx}{(x^2+1)^2}}_{I_2}$$

Лема:

$$I_{n-1} = \int \frac{dx}{(x^2+1)^{n-1}} = \left\lceil \begin{matrix} u = (x^2+1)^{1-n} \to du = -(n-1)(x^2+1)^{1-n-1} \cdot 2x\,dx \\ dv = dx \to v = x \end{matrix} \right\rfloor =$$

$$= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\int \frac{x^{2\,+1-1}\,dx}{(x^2+1)^n} = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\left(I_{n-1} - I_n\right)$$

$$I_n = \frac{1}{2(n-1)}\left(\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right)$$

$$I_2 = \frac{1}{2}\cdot\left(\frac{x}{x^2+1} + I_1\right) = \frac{1}{2}\left(\frac{x}{x^2+1} + \int\frac{dx}{x^2+1}\right) = \frac{1}{2}\left(\frac{x}{x^2+1} + \operatorname{arctg} x\right)$$

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1) $\int \frac{dx}{x-a} = \ln|x-a| + C$

2) $\int \frac{dx}{(x-a)^n} = \frac{1}{(1-n)}\cdot\frac{1}{(x-a)^{n-1}} + C$

3) $\int \frac{x\,dx}{(x^2+a^2)^n} = \lceil t = x^2+a^2 \rfloor$

4) $\int \frac{dx}{(x^2+a^2)^n} \longrightarrow$ лема ...