Неодређен интеграл

$f : (a,b) \to \mathbb{R}$

$F : (a,b) \to \mathbb{R}$ је примитивна фја фје f

ако $F \in D(a,b)$ и $F'(x) = f(x)$, $x \in (a,b)$.

• F и G примитивне фје за f онда $F(x) - G(x) = \text{const}$ за $x \in (a,b)$.

Д: $(F-G)'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$

$\underset{\text{Ферма}}{\Rightarrow} (F-G)(x) = \text{const.} = C$ $\square$

$$\int \underbrace{f(x)}_{\text{подинтегрална фја}} dx = \{ F(x) + C \ : \ C \in \mathbb{R},\ F \text{ нека примитивна фја} \}$$

Особине:

1° $\int f'(x)dx = f(x) + C$

2° $\left(\int f(x)dx \right)' = f(x)$

3° $\int f(x) + g(x)\, dx = \int f(x)dx + \int g(x)dx$

4° $\int c f(x) dx = c \int f(x) dx$, $c \neq 0$

5° $(f \cdot g)' = f'g + f \cdot g' \ / \int$

• парцијална интеграција

$$\int f(x) g'(x) dx = f(x)g(x) - \int f'(x) g(x) dx$$

6° $A \xrightarrow{g} B \xrightarrow{f} \mathbb{R}$

$(F(g(x)))' = F'(g(x)) \cdot g'(x)$

• смена променљиве: $\int f(g(x)) \cdot g'(x) dx = F(g(x)) + C$

* $\int f\,dg = fg - \int g\,df$

$$\boxed{d(g(x)) = g'(x)\,dx = dg}$$

Таблица интеграла:

1° $\int x^a\,dx = \frac{x^{a+1}}{a+1} + c$, $a \neq -1$

2° $\int \frac{dx}{x} = \ln|x| + c$

3° $\int e^x\,dx = e^x + c$

4° $\int a^x\,dx = \frac{a^x}{\ln a} + c$, $a > 0$, $a \neq 1$

5° $\int \cos x\,dx = \sin x + C$

6° $\int \sin x\,dx = -\cos x + c$

7° $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$

8° $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$

9° $\int \frac{dx}{1+x^2} = \operatorname{arctg} x + c$

10° $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

11° $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln|x + \sqrt{x^2 \pm 1}| + c$

12° $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$

① $\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx = \int \frac{x^2+1}{x^2+1}\,dx - \int \frac{dx}{x^2+1} =$

$= \int dx - \int \frac{dx}{x^2+1} = x + C_1 - \operatorname{arctg} x - C_2 =$

$= x - \operatorname{arctg} x + C$

② $I = \int \frac{x+1}{x^2+1}\,dx = \underbrace{\int \frac{x\,dx}{x^2+1}}_{I_1} + \underbrace{\int \frac{dx}{x^2+1}}_{I_2}$

$I_1 = \frac{1}{2}\int \frac{\overbrace{2x\,dx}^{=dt}}{x^2+1} =$

смена променљиве

$g(x) = x^2+1 = t \to$ нова променљиве

$g'(x)\,dx = 2x\,dx = dt$

$f(t) = \frac{1}{t}$

$=$

$= \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + c_1 = \frac{1}{2}\ln|x^2+1| + c_1 = \frac{1}{2}\ln(x^2+1) + c_1.$

$I = \frac{1}{2}\ln(x^2+1) + c_1 + \operatorname{arctg} x + c_2 = \frac{1}{2}\ln(x^2+1) + \operatorname{arctg} x + c$

(3) $\int \frac{dx}{x^2+a^2} = \frac{a}{a^2}\int \frac{\frac{1}{a}\,dx}{\left(\frac{x}{a}\right)^2+1} = \left[\begin{matrix} t = g(x) = \frac{x}{a} \\ dt = \frac{dx}{a}\end{matrix}\right] = \frac{1}{a}\int \frac{dt}{t^2+1} = \frac{1}{a}\operatorname{arctg} t + c$

$= \frac{1}{a}\operatorname{arctg}\frac{x}{a} + c$

(4) $\int \frac{dx}{x^2-a^2} = \frac{1}{a^2}\cdot a\int \frac{\frac{1}{a}\,dx}{\left(\frac{x}{a}\right)^2-1} = \left[\begin{matrix} t = \frac{x}{a} \\ dt = \frac{dx}{a}\end{matrix}\right] = -\frac{1}{a}\int \frac{dt}{1-t^2} =$

$\overset{12^\circ}{=} -\frac{1}{a}\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + c = -\frac{1}{2a}\ln\left|\frac{1+\frac{x}{a}}{1-x/a}\right| + c = \frac{1}{2a}\ln\left|\frac{a-x}{a+x}\right| + c$

(5) $\int \frac{\cos x+\sin x}{\sqrt[3]{\sin x-\cos x}}\,dx = \left[\begin{matrix} t = \sin x - \cos x \\ dt = (\cos x + \sin x)\,dx\end{matrix}\right] = \int \frac{dt}{\sqrt[3]{t}} = \int t^{-1/3}\,dt$

$= \frac{t^{-1/3+1}}{-1/3+1} + C = \frac{t^{2/3}}{2/3} + c = \frac{3}{2}(\sin x - \cos x)^{2/3} + c.$

(6) $\int \frac{dx}{e^x+e^{-x}}\cdot\frac{e^x}{e^x} = \int \frac{\overbrace{e^x\,dx}^{=dt}}{\underbrace{e^{2x}+1}_{t^2+1}} = \left[\begin{matrix} t = e^x \\ dt = e^x\,dx\end{matrix}\right] = \int \frac{dt}{t^2+1} =$

$= \operatorname{arctg} t + c = \operatorname{arctg} e^x + c.$

(7) $\int \frac{dx}{(e^x+e^{-x})^2} = \int \frac{dx}{e^{2x}+2+e^{-2x}}\cdot\frac{e^{2x}}{e^{2x}} = \frac{1}{2}\int \frac{2e^{2x}\,dx}{(e^{2x})^2+2e^{2x}+1} = \left[\begin{matrix} t = e^{2x} \\ dt = 2e^{2x}\,dx\end{matrix}\right]$

$= \frac{1}{2}\int \frac{dt}{t^2+2t+1} = \frac{1}{2}\int \frac{dt}{(t+1)^2} = \left[\begin{matrix} u = t+1 \\ du = dt\end{matrix}\right] = \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\int u^{-2}\,du$

$= \frac{1}{2}\frac{u^{-1}}{-1} + c = -\frac{1}{2}\frac{1}{t+1} + c = -\frac{1}{2}\frac{1}{e^{2x}+1} + c$

(8) $\int \frac{dx}{\sqrt{x(1-x)}} = \left[x \in (0,1)\right] = 2\int \frac{\overbrace{\frac{dx}{2\sqrt{x}}}^{dt}}{\sqrt{1-x}} = \left[\begin{matrix} t = \sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}\,dx \\ x = t^2\end{matrix}\right]$

$= 2\int \frac{dt}{\sqrt{1-t^2}} = 2\arcsin t + c = 2\arcsin\sqrt{x} + c$

(9) $\int \frac{dx}{\cos^2 x(\operatorname{tg}^2 x+3)} = \left[\begin{matrix} t = \operatorname{tg} x \\ dt = \frac{dx}{\cos^2 x}\end{matrix}\right] = \int \frac{dt}{t^2+3} \overset{(3)\ a=\sqrt{3}}{=} \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}} + c$

$= \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{\operatorname{tg} x}{\sqrt{3}} + c$

(10) $\displaystyle\int \frac{\overset{dt}{dx}}{x(\ln^2 2x+1)} = \left\lceil \begin{array}{l} t=\ln 2x \\ dt = \frac{dx}{2x}\cdot 2 \end{array} \right\rfloor = \int \frac{dt}{t^2+1} = \operatorname{arctg} t + C$

$= \operatorname{arctg}(\ln 2x) + C$

(11) $\displaystyle\frac{1}{2}\int \frac{2x\,dx}{x^4+4} = \left\lceil \begin{array}{l} t=x^2 \\ dt=2x\,dx \end{array} \right\rfloor = \frac{1}{2}\int \frac{dt}{t^2+4} \overset{(3)\ a=2}{=} \frac{1}{4}\operatorname{arctg}\frac{t}{2} + C$

$= \frac{1}{4}\operatorname{arctg}\frac{x^2}{2} + C.$

(12) $\displaystyle\int \frac{dx}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x} = \int \frac{(1+\sin x)\,dx}{1-\sin^2 x} = \int \frac{(1+\sin x)\,dx}{\cos^2 x}$

$\displaystyle = \int \frac{dx}{\cos^2 x} + \underbrace{\int \frac{\sin x\,dx}{\cos^2 x}}_{I_2} = \operatorname{tg} x + \frac{1}{\cos x} + C.$

$\displaystyle I_2 = \int \frac{\sin x\,dx}{\cos^2 x} = \left\lceil \begin{array}{l} t=\cos x \\ dt=-\sin x\,dx \end{array} \right\rfloor = -\int \frac{dt}{t^2} = \frac{1}{t} + C_1 =$

$\displaystyle = \frac{1}{\cos x} + C_1$

(13) $\displaystyle\int \sin 5x \cos 7x\,dx$

(14) $\displaystyle\int \left(x^3 + \frac{1}{\sqrt{x}}\right)^2 dx \rightarrow$ за вежбу

(15) $\displaystyle\int \ln x\,dx = \left\lceil \begin{array}{ll} u=\ln x & \rightarrow u'(x)\,dx = du = \frac{1}{x}\,dx \\ dv = v'(x)\,dx = dx & \rightarrow v(x) = \int dv = \int v'(x)\,dx = \int dx \\ & \quad = x \end{array} \right\rfloor$

ПИ $\displaystyle\int u\,dv = u\cdot v - \int v\,du$

$\displaystyle = \ln x\cdot x - \int x\cdot\frac{1}{x}\,dx = \ln x\cdot x - \int dx = \ln x\cdot x - x + C.$

(16) $\displaystyle\int \underbrace{\operatorname{arctg} x}_{u}\,dx = \left\lceil \begin{array}{ll} u=\operatorname{arctg} x & \rightarrow du = u'(x)\,dx = \frac{1}{x^2+1}\,dx \\ dv = dx & \rightarrow v = x \end{array} \right\rfloor =$

$\displaystyle = x\cdot\operatorname{arctg} x - \underbrace{\int \frac{x}{x^2+1}\,dx}_{I_1} \overset{(2)}{=} x\cdot\operatorname{arctg} x - \frac{1}{2}\ln(x^2+1) + C.$

(17) $I_1 = \int e^{ax} \sin bx \, dx \quad , \; a, b \in \mathbb{R} \setminus \{0\}$

$I_2 = \int e^{ax} \cos bx \, dx$

$$I_1 = \int e^{ax} \sin bx \, dx = \begin{bmatrix} u = \sin bx & \to du = b \cos bx \, dx \\ dv = e^{ax} dx & \to v = \int e^{ax} dx = \begin{bmatrix} t = ax \\ dt = a \cdot dx \end{bmatrix} = \frac{1}{a} e^{t} = \frac{1}{a} e^{ax} \end{bmatrix}$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \underbrace{\int e^{ax} \cos bx \, dx}_{I_2}$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} I_2 + C_1$$

$$I_2 = \int e^{ax} \cos bx \, dx = \begin{bmatrix} u = \cos bx & , \; du = -b \sin bx \, dx \\ dv = e^{ax} dx & \to v = \frac{1}{a} e^{ax} \end{bmatrix}$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \underbrace{\int e^{ax} \sin bx \, dx}_{I_1} = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} I_1 + C_2$$

$$I_1 = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \left(\frac{1}{a} e^{ax} \cos bx + \frac{b}{a} I_1 \right) + C$$

$$\left(1 + \frac{b^2}{a^2}\right) I_1 = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx + C$$

$$I_1 = \frac{a e^{ax} \sin bx - b e^{ax} \cos bx}{a^2 + b^2} + \underbrace{\frac{a^2}{a^2+b^2} \cdot C}_{D \in \mathbb{R}}$$

$$I_2 = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \cdot \frac{a e^{ax} \sin bx - b e^{ax} \cos bx}{a^2 + b^2} + \frac{b}{a} D + C_2$$

$$= \frac{a e^{ax} \cos bx + b e^{ax} \sin bx}{a^2 + b^2} + E$$