① $f(x) = x\sqrt{x^2-2x}$ → Испитати ток и скицирати график ф-је

1° $D_f = ?$

$$x^2-2x \geq 0$$
$$x(x-2) \geq 0 \implies D_f = (-\infty, 0] \cup [2, +\infty)$$

Нуле ф-је: 0 и 2, $f(0)=0$, $f(2)=0$

знак: $f > 0,\ x > 2$

$f < 0,\ x < 0$

f није парна, ни непарна

f није периодична

2° асимптоте

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \sqrt{x^2-2x} = +\infty \implies f \text{ нема ни косу, ни хоризонталну}$$

$$f(x) = x\sqrt{x^2-2x} = x|x|\underbrace{\sqrt{1-\frac{2}{x}}}_{\to 1,\ x\to\pm\infty} \sim \operatorname{sgn} x \cdot x^2, \quad x \to \pm\infty$$

3° монотоност

$$f'(x) = \left(x\sqrt{x^2-2x}\right)' = \sqrt{x^2-2x} + x \cdot \frac{1}{2\sqrt{x^2-2x}} \cdot (2x-2) =$$

$$= \frac{x^2-2x + x\cdot(x-1)}{\sqrt{x^2-2x}} = \frac{2x^2-3x}{\sqrt{x^2-2x}} = \frac{x\cdot(2x-3)}{\sqrt{x^2-2x}}$$

$f'\left(\frac{3}{2}\right) = 0,\ \frac{3}{2} \notin D_f$

		0		$3/2$		2	
f'	+	0					+
f	↗						↗

$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{h\sqrt{h^2-2h} - 0}{h} = \lim_{h\to 0^-} \sqrt{h^2-2h} = 0$$

$$f'_+(2) = \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^+} \frac{(2+h)\sqrt{(2+h)^2 - 2(2+h)} - 0}{h} = \lim_{h\to 0^+} \frac{(2+h)\sqrt{4+4h+h^2-4-2h}}{h}$$

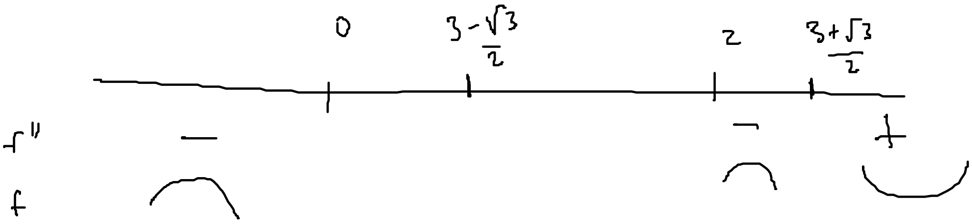
$$= \lim_{h\to 0^+} \frac{(2+h)\sqrt{2h+h^2}}{h} = \lim_{h\to 0^+} \underbrace{(2+h)}_{\downarrow\ 2} \underbrace{\sqrt{\frac{2}{h}+1}}_{\downarrow\ +\infty} = +\infty$$

4° конвексност:

$$f''(x) = \left(\frac{2x^2-3x}{\sqrt{x^2-2x}}\right)' = \frac{(4x-3)\sqrt{x^2-2x} - (2x^2-3x)\cdot\frac{1}{2\sqrt{x^2-2x}}\cdot(2x-2)}{x^2-2x}$$

$$= \frac{(4x-3)(x^2-2x) - (2x^2-3x)\cdot(x-1)}{(x^2-2x)^{3/2}} = \frac{4x^3-11x^2+6x-2x^3+5x^2-3x}{(x^2-2x)^{3/2}}$$

$$= \frac{2x^3-6x^2+3x}{(x^2-2x)^{3/2}} = \frac{x\cdot(2x^2-6x+3)}{(x^2-2x)^{3/2}} \rightsquigarrow x_{1/2} = \frac{6\pm\sqrt{36-24}}{4} = \frac{6\pm\sqrt{12}}{4} = \frac{3\pm\sqrt{3}}{2}$$



$$\to f\left(\frac{3+\sqrt{3}}{2}\right) = \frac{3+\sqrt{3}}{2}\sqrt{\left(\frac{3+\sqrt{3}}{2}\right)\cdot\underbrace{\left(\frac{3+\sqrt{3}}{2}-2\right)}_{=\frac{\sqrt{3}-1}{2}}}$$

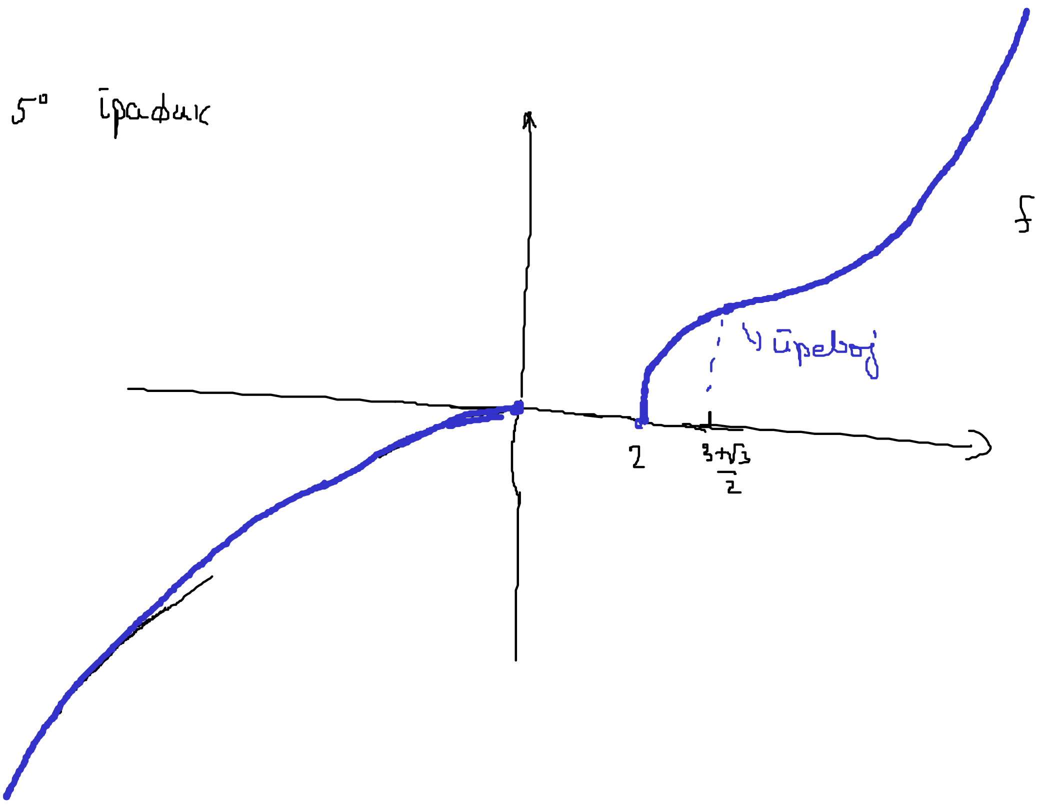
$$= \frac{3+\sqrt{3}}{4}\sqrt{(3+\sqrt{3})(\sqrt{3}-1)}$$

$$3\sqrt{3}-\sqrt{3}+3-\sqrt{3}$$

$$\sqrt{3+\sqrt{3}}$$

$$= \frac{(3+\sqrt{3})^{3/2}}{4}$$

5° график



(2) $f, g : [a,b] \to \mathbb{R}$, $f, g \in C[a,b] \cap C^1(a,b)$

$f', g' > 0$ ↗ на (a,b)

$$\overset{?}{\Rightarrow} \exists c \in (a,b) \quad \frac{f(b)-f(a)}{b-a}\cdot\frac{g(b)-g(a)}{b-a} = \underline{f'(c)\cdot g'(c)}$$

Лагранж $\Rightarrow$ $\exists d_1 \in (a,b)$; $\frac{f(b)-f(a)}{b-a} = f'(d_1)$

$f \in C[a,b] \cap D(a,b)$

Лагранж $\Rightarrow$ $\exists d_2 \in (a,b)$ $\frac{g(b)-g(a)}{b-a} = g'(d_2)$

$g \in C[a,b] \cap D(a,b)$

$? \ \exists c \in (a,b) \quad f'(d_1)\cdot g'(d_2) = f'(c)\, g'(c) \ ?$

БУО $d_1 \le d_2$

Ако $d_1 = d_2 \Rightarrow c = d_1 = d_2$.

Ако $d_1 < d_2$ $\quad f'(d_1)g'(d_1) \le \underbrace{f'(d_1)\cdot g'(d_2)}_{= A} \le f'(d_2)\cdot g'(d_2)$

$g'(d_2) > 0$

$f'(d_1) \le f'(d_2)$

$F(x) = f'(x)\, g'(x)$, $F \in C[d_1, d_2]$ $\Big\}$ $\xRightarrow{\text{Коши-Болцано}}$ $\exists c \in (d_1, d_2)$ $F(c) = A$.

$A \in [F(d_1), F(d_2)]$

$F(c) = f'(c)\cdot g'(c) = f'(d_1)g'(d_2) = \frac{f(b)-f(a)}{b-a}\ \frac{g(b)-g(a)}{b-a}$.

③ У зависности од $x \in \mathbb{R}$, одредити $\lim\limits_{n\to\infty} a_n$, ако је

$a_n = \left(\frac{\log n^x}{n+1} + x^n + n 2^n \right)^{1/n} = \sqrt[n]{x \cdot \frac{\log n}{n+1} + x^n + n\cdot 2^n}$

1° $-2 \le x \le 2 \Rightarrow \underbrace{n\cdot 2^n - 2^n - 2^n}_{= (n-2)\,2^n} \le x \frac{\log n}{n+1} + x^n + n\cdot 2^n \le 2^n + 2^n + n\cdot 2^n = (n+2)\cdot 2^n$

$\frac{\log n}{n+1} \xrightarrow{n\to\infty} 0$

$2\sqrt[n]{n-2} \le a_n \le 2\sqrt[n]{n+2}$

$2\sqrt[n]{n-2} \to 2$, $\quad a_n \xrightarrow{\text{То 2П}} 2$, $\quad 2\sqrt[n]{n+2} \to 2$

2° $x > 2$ $\qquad x^n \boxed{>} n\cdot 2^n$, $n \ge n_0$ $\ /: 2^n$

$\left(\frac{x}{2}\right)^n \boxed{>} n$, $n \ge n_0$

$\sqrt[n]{x^n} \le a_n \le \sqrt[n]{3x^n}$

$\sqrt[n]{x^n} = x$, $\quad a_n \xrightarrow{\text{То 2П}} x$, $\quad \sqrt[n]{3x^n} \to x$

$3^\circ \quad x<-2$

$$a_n = \sqrt[n]{x\cdot\frac{\log n}{n+2} + x^n + n2^n}$$

n - парно $\quad x^n \geq n2^n$

$$x\cdot\frac{\log n}{n+2} \longrightarrow 0, \ n\to+\infty \Rightarrow \left| x\cdot\frac{\log n}{n+2}\right| < 1 \ \text{за } n\geq n_0$$

$$\sqrt[n]{x^n} \leq a_n \leq \sqrt[n]{3x^n}, \quad \underline{\underline{a_n \to |x|}}, \quad n=2k, \ k\to+\infty$$

n - непарно $\quad x^n = -|x|^n \qquad \dfrac{x^n+n\cdot 2^n}{\frac{1}{2}x^n} \to 2, \quad n\to+\infty$

$|x|^n \geq n\cdot 2^n \qquad n2^n < \frac{1}{2}x^n$

$-|x|^n + n\cdot 2^n \sim -|x|^n, \quad x<-2, n\to+\infty$

за n довољно велико и непарно

$$\sqrt[n]{-1+\tfrac{1}{2}x^n} \leq a_n \leq \sqrt[n]{1+x^n+\tfrac{1}{2}x^n}$$

$$\downarrow \qquad\quad \downarrow \qquad\quad \downarrow$$

$$x \qquad\quad x \qquad\quad x$$

$$n=2k+1, \quad \underline{\underline{a_n \to x}}, \quad k\to+\infty$$

$\Rightarrow a_n$ не конвергира, $n\to+\infty$

$1^\circ \ |x|\leq 2 \Rightarrow \ a_n \to 2$

$2^\circ \ x>2 \Rightarrow a_n \to x$

$3^\circ \ x<-2 \Rightarrow a_n$ дивергира.

(4) $f(x) = \dfrac{x^2-2x+2}{x^2-3}\operatorname{arctg}|x-2|$

а) Испитати диф. фје

б) Испитати равн неп. фје. на $(0,1)$ и $(e,+\infty)$

$$D_f = \mathbb{R}\setminus\{-\sqrt{3},\sqrt{3}\}$$

$$f'(x) = \frac{(2x-2)(x^2-3) - 2x(x^2-2x+2)}{(x^2-3)^2}\cdot\operatorname{arctg}|x-2|$$

$$+\ \frac{x^2-2x+2}{x^2-3}\ \frac{1}{1+|x-2|^2}\cdot\operatorname{sgn}(x-2), \quad x\neq 2$$

$\Rightarrow f$ диф на $\mathbb{R}\setminus\{-\sqrt{3},\sqrt{3},2\}$

Шта се дешава у $x=2$?

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-\overset{=0}{f(2)}}{h}=\lim_{h\to 0}\frac{\frac{(2+h)^2-2(2+h)+2}{(2+h)^2-3}\cdot \operatorname{arctg}|2+h-2|-0}{h}$$

$$=\lim_{h\to 0}\frac{h^2+4h+\cancel{4}-\cancel{4}-2h+2}{h^2+4h+4-3}\cdot\frac{\overset{\sim |h|+o(h)}{\operatorname{arctg}|h|}}{h}$$

$$=\lim_{h\to 0}\underbrace{\frac{h^2+2h+2}{h^2+4h+1}}_{\to 2}\cdot\left(\underset{\pm 1,\ h\gtrless 0}{\operatorname{sgn} h}+o(1)\right)$$ $\Rightarrow$ лимес не постоји $\Rightarrow f$ није диф у 2!

б) $f(x)=\frac{x^2-2x+2}{x^2-3}\operatorname{arctg}|x-2|$

$(0,1)$: f диф на $[0,1]$ $\Rightarrow f$ непр. на $[0,1]$ $\overset{\text{Кантор}}{\Rightarrow}$ f равн непр на $[0,1]$

$\Rightarrow f$ равн непр на $(0,1)$

$(e,+\infty)$: f непр на $[e,+\infty)$ $\overset{\text{Кантор}}{\Rightarrow}$ f равн непр на $[e,M]$, $M>e$.

$\lim\limits_{x\to+\infty}f(x)=\pi/2\in\mathbb{R}\Rightarrow f$ равн непр на $[M,+\infty)$ за неко $M>0$

$\Rightarrow f$ равн непр на $[e,+\infty)$

5. а) Наћи Маклоренов полином за $f(x)=\arcsin x$ до 3. степена

б) $$\lim_{n\to\infty}\underbrace{\frac{n\left(e^{\arcsin\frac{1}{n}}-e^{\operatorname{arctg}\frac{n}{n^2+1}}\right)}{n\cos\frac{1}{n}\sin\frac{2}{n}-2}}_{b_n}$$

а) $f'(x)=\frac{1}{\sqrt{1-x^2}}\Rightarrow f'(0)=1$

$f''(x)=-\frac{1}{2(1-x^2)^{3/2}}(-2x)=x\cdot(1-x^2)^{-3/2}$ $\quad f''(0)=0$

$$f'''(x) = (1-x^2)^{-3/2} - 3/2\,(1-x^2)^{-5/2}\cdot(-2x^2) \qquad f'''(0) = 1$$

$$\arcsin x = \underset{0}{\underset{\parallel}{\arcsin 0}} + 1\cdot\frac{x}{1!} + \frac{x^3}{3!} + o(x^3) = x + \frac{x^3}{6} + o(x^3), \; x\to 0$$

б)
$$\operatorname{arctg} x = x - \frac{x^3}{3} + o(x^3), \; x\to 0 \qquad f' = \frac{1}{1+x^2}$$

$$f'' = -\frac{1}{(1+x^2)^2}2x$$

$$f''' = -\frac{2}{3(1+x^2)^2} + \cdots x$$

$$e^{\arcsin\frac{1}{n}} = e^{\frac{1}{n}+\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)} = 1 + \left(\frac{1}{n}+\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)\right)$$

а) $x = \frac{1}{n}$

$$+ \frac{\left(\frac{1}{n}+\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)\right)^2}{2} + \frac{\left(\frac{1}{n}+\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)\right)^3}{6} + \underset{o\left(\frac{1}{n^3}\right)}{\underset{\parallel}{o\left(\left(\frac{1}{n}+\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)\right)^3\right)}}$$

$$= 1 + \frac{1}{n} + \frac{1}{6n^3} + \frac{1}{2n^2} + \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)$$

$$\frac{n}{n^2+1} = n\cdot(n^2+1)^{-1} = n\cdot n^{-2}\left(1+\frac{1}{n^2}\right)^{-1} = \frac{1}{n}\cdot\left(1-\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right) = \frac{1}{n} - \frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$$

$$e^{\operatorname{arctg}\frac{n}{n^2+1}} = e^{\operatorname{arctg}\left(\frac{1}{n}-\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)\right)} = e^{\left(\frac{1}{n}-\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)\right) - \frac{1}{3}\left(\frac{1}{n}-\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)\right)^3 + o\left(\frac{1}{n^3}\right)}$$

$$= e^{\frac{1}{n}-\frac{1}{n^3}-\frac{1}{3n^3}+o\left(\frac{1}{n^3}\right)} = e^{\frac{1}{n}-\frac{4}{3n^3}+o\left(\frac{1}{n^3}\right)} =$$

$$= 1 + \frac{1}{n} - \frac{4}{3n^3} + o\left(\frac{1}{n^3}\right) + \frac{1}{2}\left(\frac{1}{n}-\frac{4}{3n^3}+o\left(\frac{1}{n^3}\right)\right)^2 + \frac{1}{6}\left(\frac{1}{n}-\frac{4}{3n^3}+o\left(\frac{1}{n^3}\right)\right)^3 + o\left(\frac{1}{n^3}\right)$$

$$= 1 + \frac{1}{n} + \frac{1}{2n^2} - \frac{4}{3n^3} + \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)$$

$$= 1 + \frac{1}{n} + \frac{1}{2n^2} - \frac{7}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$$

$$\cos\frac{1}{n} = 1 - \frac{1}{2}\frac{1}{n^2} + o\left(\frac{1}{n^3}\right)$$

$$\sin\frac{2}{n} = \frac{2}{n} - \frac{2}{6n^3} + o\left(\frac{1}{n^4}\right) = \frac{1}{n}\left(2 - \frac{1}{3n^2} + o\left(\frac{1}{n^3}\right)\right)$$

$$n\cos\frac{1}{n}\sin\frac{2}{n} = \cancel{n}\cdot\frac{1}{\cancel{n}}\left(2-\frac{1}{3n^2}+o\left(\frac{1}{n^3}\right)\right)\left(1-\frac{1}{2n^2}+o\left(\frac{1}{n^3}\right)\right) = 2 - \frac{4}{3n^2} + o\left(\frac{1}{n^3}\right)$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n\left(\cancel{1}+\cancel{\frac{1}{n}}+\cancel{\frac{1}{2n^2}}+\frac{1}{3n^3}+o\left(\frac{1}{n^3}\right)-\left(\cancel{1}+\cancel{\frac{1}{n}}+\cancel{\frac{1}{2n^2}}-\frac{7}{6}\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)\right)\right)}{2-\frac{4}{3n^2}+o\left(\frac{1}{n^3}\right)-2}$$

$$= \lim_{n\to\infty} \frac{\frac{9}{6}\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)}{-\frac{4}{3}\frac{1}{n^2}+o\left(\frac{1}{n^3}\right)} = \frac{\frac{3}{2}}{-4/3} = -\frac{9}{8}.$$