① $f(x) = -\frac{1}{|x|} + \operatorname{arctg}\frac{2x}{x^2-1}$

1° $D_f = (-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,+\infty)$

$|x| \neq 0$

$x^2-1 \neq 0$

парност:

$f(-x) = -\frac{1}{|-x|} + \operatorname{arctg}\frac{2(-x)}{(-x)^2-1} = -\frac{1}{|x|} - \operatorname{arctg}\frac{2x}{x^2-1} \neq f(x)$

$\Rightarrow$ f није ни парна, ни непарна

f није периодична

$f(a)=0 \Leftrightarrow -\frac{1}{|a|} + \operatorname{arctg}\frac{2a}{a^2-1} = 0 \Leftrightarrow \operatorname{arctg}\frac{2a}{a^2-1} = \frac{1}{|a|}$ → касније ћемо нуле и знак

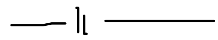
2° асимптоте:

$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to-1^-} \underbrace{-\frac{1}{|x|}}_{\to -1} + \underbrace{\operatorname{arctg}\underbrace{\frac{2x}{x^2-1}}_{\to -\infty}}_{\to -\frac{\pi}{2}} = -1-\frac{\pi}{2} = f_-(-1)$

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to-1^+}\left(\underbrace{-\frac{1}{|x|}}_{\to -1} + \operatorname{arctg}\underbrace{\frac{2x}{x^2-1}}_{+\infty}\right) = -1+\frac{\pi}{2} > 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\left(\underbrace{-\frac{1}{|x|}}_{\to -\infty} + \underbrace{\operatorname{arctg}\frac{2x}{x^2-1}}_{\to 0}\right) = -\infty \Rightarrow$ вертикална асимптота у 0 слева

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\left(-\frac{1}{|x|} + \operatorname{arctg}\frac{2x}{x^2-1}\right) = -\infty \Rightarrow$



у 0 здесна

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}\left(\underbrace{-\frac{1}{|x|}}_{-1} + \underbrace{\operatorname{arctg}\underbrace{\frac{2x}{x^2-1}}_{0^-}}_{\to -\frac{\pi}{2}}\right) = -1-\frac{\pi}{2} < 0$

$\lim\limits_{x\to 1^+} f(x) = -1+\frac{\pi}{2} > 0$

$\lim\limits_{x\to \pm\infty} f(x) = \lim\limits_{x\to\pm\infty}\left(\underbrace{-\frac{1}{|x|}}_{\to 0} + \operatorname{arctg}\underbrace{\frac{2x}{x^2-1}}_{\to 0}\right) = 0$ → хоризонтална асимптота

3° монотоност $\quad |x| = \operatorname{sgn} x \cdot x \rightarrow$ ово је диф у $\mathbb{R}\setminus\{0\}$

$$f'(x) = \left(-\frac{1}{\operatorname{sgn} x \cdot x} + \operatorname{arctg} \frac{2x}{x^2-1} \right)' =$$

$$= \frac{1}{\operatorname{sgn} x} \frac{1}{x^2} + \frac{1}{1+\frac{4x^2}{(x^2-1)^2}} \cdot \frac{2(x^2-1) - 2x\cdot 2x}{(x^2-1)^2} =$$

$$= \operatorname{sgn} x \frac{1}{x^2} + \frac{-2-2x^2}{x^4-2x^2+1+4x^2} = \operatorname{sgn} x \cdot \frac{1}{x^2} - \frac{2(x^2+1)}{(x^2+1)^2} =$$

$$= \operatorname{sgn} x \cdot \frac{1}{x^2} - \frac{2}{x^2+1} = \frac{\operatorname{sgn} x \cdot (x^2+1) - 2x^2}{x^2 \cdot (x^2+1)} = \begin{cases} \frac{1-x^2}{x^2(x^2+1)} & x>0 \\ \frac{-3x^2-1}{x^2(x^2+1)} & x<0 \end{cases}$$

		-1		0		1	
f'	$-$		$-$		$+$		$-$
	↘		↘		↗		↘

$\Rightarrow f$ нема локалне екстремуме јер није деф у 1.

$(1-h)^{-1} = 1+h+o(h)$

$$f'_-(-1) = \lim_{h\to 0^-} \frac{f(-1+h) - f(-1)}{h} = \frac{-\frac{1}{1-h} + \operatorname{arctg}\frac{2(-1+h)}{(-1+h)^2-1} - \left(-1-\frac{\pi}{2}\right)}{h} =$$

$$= -1+h+o(h) + \ldots$$

$$\operatorname{arctg}\frac{-2(1-h)}{-2h+h^2} = \operatorname{arctg}\frac{2(1-h)}{\underbrace{-2h+h^2}_{\to -\infty}} \overset{*}{=} -\frac{\pi}{2} - \left(-\frac{-2h+h^2}{2(1-h)}\right) + o(h) = -\frac{\pi}{2} - \frac{2h-h^2}{2(1-h)} + o(h)$$

$$\operatorname{arctg} x = \ ? \qquad x \to +\infty, \ \to -\infty$$

$$(\operatorname{arctg} x)' = \frac{1}{1+x^2}, \quad \left(\operatorname{arctg}\frac{1}{x}\right)' = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+1}$$

$$\left(\operatorname{arctg} x + \operatorname{arctg}\frac{1}{x}\right)' = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

$f(x) = \operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} \rightarrow$ константна на $(0,+\infty)$ јер $\frac{1}{x}$ није деф у 0

$= const$

$\underline{x=1}$ $\quad f(1) = const = \operatorname{arctg} 1 + \operatorname{arctg} 1 = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$

$x>0$ $\quad \operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2} \Rightarrow \operatorname{arctg} x = \frac{\pi}{2} - \operatorname{arctg}\frac{1}{x}$

$x\to+\infty \Rightarrow \frac{1}{x}\to 0 \qquad \operatorname{arctg}\frac{1}{x} = \frac{1}{x} - \frac{1}{3x^3} + o\left(\frac{1}{x^3}\right) = \frac{1}{x} + o\left(\frac{1}{x^2}\right)$

$$\Rightarrow \operatorname{arctg} x = \frac{\pi}{2} - \frac{1}{x} + o\left(\frac{1}{x^2}\right), \quad x\to+\infty$$

$$\operatorname{arctg} x = -\operatorname{arctg}(\underset{\downarrow\, +\infty}{-x}) = -\left(\frac{\pi}{2} - \frac{1}{-x} + o\left(\frac{1}{x^2}\right)\right) \qquad x\to-\infty$$

$\frac{1}{1-h} = 1+h+o(h)$

$$\overset{*}{=} -\frac{\pi}{2} - \frac{1}{x} + o\left(\frac{1}{x^2}\right), \quad x\to-\infty$$

$$f'_-(-1) = \lim_{h\to 0^-} \frac{-1-h+o(h) - \cancel{\frac{\pi}{2}} - \frac{2h-h^2}{2(1-h)} + o(h) - \left(-\cancel{1} - \cancel{\frac{\pi}{2}}\right)}{h} =$$

$$= \lim_{h\to 0^-} \frac{-h - h\cdot\left(\frac{2-h}{2(1-h)}\right) + o(h)}{h} = \lim_{h\to 0^-} -1 - \underbrace{\frac{2-h}{2(h+1)}}_{\downarrow\, 1} + o(1) = -2$$

$f'_+(-1) = -2$

$f'_+(1) = 0$ $\Big\}$ расписати за вежбу

$f'_-(1) = 0$

4° конвексност

$$f''(x) = \frac{-2\operatorname{sgn} x\,(x^4+2x^2+1) + 4x^4}{(x^2+1)^2x^3} = \begin{cases} \dfrac{2x^4-4x^2-2}{(x^2+1)^2x^3} & x>0 \\[2ex] \dfrac{\overset{>0}{6x^4+4x^2+2}}{\underset{<0}{(x^2+1)^2x^3}} & x<0 \end{cases}$$

$2x^4-4x^2-2 =$

$= 2(x^4-2x^2\underset{+1-2}{-1})$

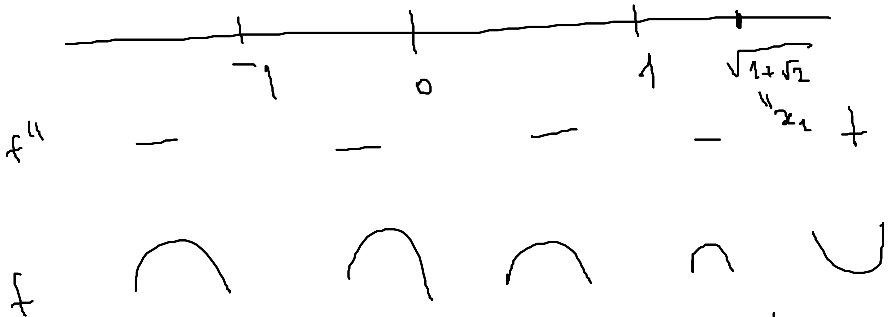
$= 2((x^2-1)^2-2)$

$= 2((x^2-1-\sqrt{2})(\underbrace{x^2-1+\sqrt{2}}_{>0}))$

$\Downarrow$

$x_{1/2} = \pm\sqrt{1+\sqrt{2}}$

$x_1 = \sqrt{1+\sqrt{2}} > 1$



$\Downarrow$

f има превојну тачку у x_1 $\quad f(x_1) = -\frac{1}{\sqrt{1+\sqrt{2}}} + \operatorname{arctg}\frac{2\sqrt{1+\sqrt{2}}}{\sqrt{2}} > 0$

* Нуле и знак ф-је f

$\lim_{x\to-\infty} f(x) = 0$

$f \downarrow$ на $(-\infty,-1)$ $\Big\} \Rightarrow f<0$ на $(-\infty,-1)$

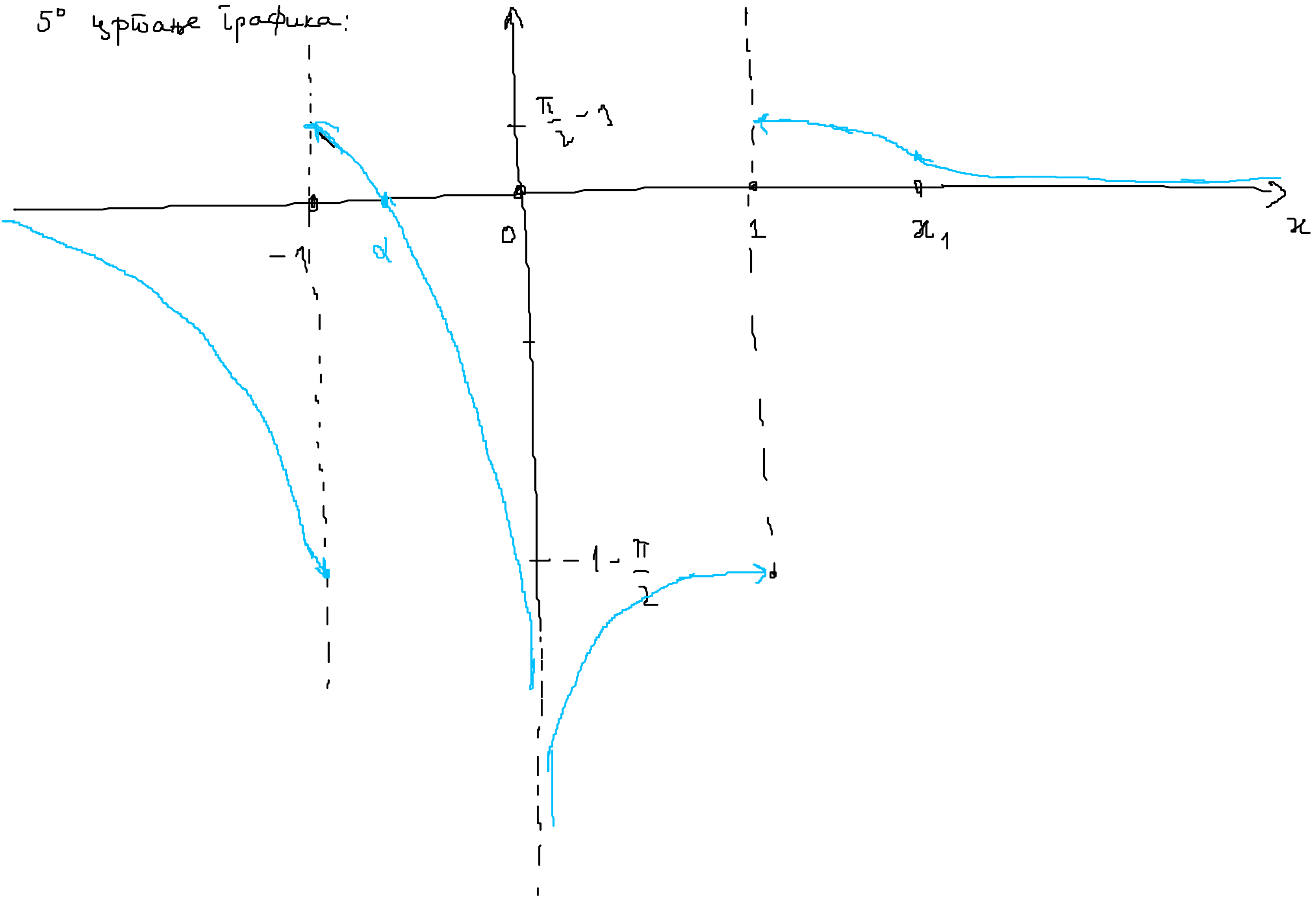
$$\left.\begin{array}{l} f_+(-1) = -1 + \frac{\pi}{2} > 0 \\ \lim\limits_{x\to 0^-} f(x) = -\infty \\ f \text{ нейр на } (-1,0) \end{array}\right\} \overset{\text{Коши-Болцано}}{\Longrightarrow} \quad \exists\, c \in (-1,0) \quad f(c)=0 .$$

$$\left.\begin{array}{l} f_-(1) = -1 - \frac{\pi}{2} < 0 \\ f \uparrow \text{ на } (0,1) \end{array}\right\} \Rightarrow \quad f < 0 \quad \text{на} \quad (0,1)$$

$$\forall x \in (0,1) \qquad f(x) < f_-(1) < 0$$

$$\left.\begin{array}{l} f_+(1) = -1 + \frac{\pi}{2} > 0 \\ f \downarrow \\ \lim\limits_{x\to +\infty} f(x) = 0 \end{array}\right\} \Rightarrow \quad f > 0 \quad \text{на} \quad (1, +\infty)$$

5° цртање графика:



② $f(x) = 2\,\mathrm{arctg}\frac{1}{|x|} + \arcsin\frac{2x}{1+x^2}$

$2\,\underbrace{\mathrm{arctg}\,1}_{=\frac{\pi}{4}} + \underbrace{\arcsin 1}_{=\frac{\pi}{2}} = \pi$

$2\,\mathrm{arctg}\,1 + \arcsin\frac{-2}{2} = 0$

1° $D_f = (-\infty, 0) \cup (0, +\infty)$

$x \neq 0$, $-1 \le \frac{2x}{1+x^2} \le 1 \Rightarrow -(1+x^2) \le 2x \le 1+x^2$ ✓ $x \in \mathbb{R}$

f није ни парно, ни непарно

f није ни периодична

нуле касније

2° асимптоте

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(2\,\mathrm{arctg}\underbrace{\frac{1}{|x|}}_{\to +\infty} + \arcsin\underbrace{\frac{2x}{1+x^2}}_{\to 0}\right) = \pi + 0 = \pi$$

$$\lim_{x\to 0^+} f(x) = \pi$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} 2\,\mathrm{arctg}\underbrace{\frac{1}{|x|}}_{\to 0} + \arcsin\underbrace{\frac{2x}{1+x^2}}_{\to 0} = 0$$

$$\lim_{x\to -\infty} f(x) = 0$$

3° монотоност

$$f'(x) = \left(2\,\mathrm{arctg}\frac{1}{\mathrm{sgn}\,x\cdot x} + \arcsin\frac{2x}{1+x^2}\right)' =$$

$$= 2\frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{\mathrm{sgn}\,x\cdot x^2}\right) + \frac{1}{\sqrt{1-\frac{4x^2}{(1+x^2)^2}}}\cdot\frac{2(1+x^2)-2x\cdot 2x}{(1+x^2)^2} =$$

$$= -\frac{2\,\mathrm{sgn}\,x}{x^2+1} + \frac{2(1-x^2)}{(1+x^2)\sqrt{(1-x^2)^2}} = -\frac{2\,\mathrm{sgn}\,x}{x^2+1} + \frac{2(1-x^2)}{(1+x^2)\cdot|1-x^2|} =$$

$$\underset{x\neq\pm 1}{=} \frac{-2\,\mathrm{sgn}\,x + 2\,\mathrm{sgn}(1-x^2)}{1+x^2} = \begin{cases} 0, & x<-1,\ x\in(0,1) \\ \frac{4}{1+x^2}, & x\in(-1,0) \\ -\frac{4}{1+x^2}, & x\in(1,+\infty) \end{cases}$$

$x<-1$: $\mathrm{sgn}\,x = -1$, $\mathrm{sgn}(1-x^2) = -1$, $-2\cdot(-1) + 2\cdot(-1)$

$f'_+(-1) = 2$, $f'_-(0) = 4$, $f'_+(1) = -2$

		-1		0		1	
f'	0		$+$		0		$-$
f	const		$\nearrow$		const		$\searrow$
$f(x)$	0		>0 π		$\pi>0$		>0 0

4° конвексност

$$f''(x) = \begin{cases} 0, & x<-1,\ x\in(0,1) \\ -\dfrac{4}{(1+x^2)^2}\cdot 2x, & x\in(-1,0) \\ \dfrac{4}{(1+x^2)^2}\cdot 2x, & x>1 \end{cases}$$

	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,+\infty)$
f''	0	+	0	+
f	const	$\cup$	const	$\cup$

5° цртање графика