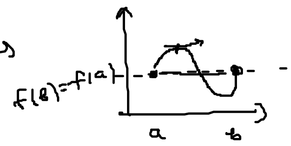
Теореме:

Ферма: f диф у a, f има локални екстремум у $a \Rightarrow f'(a)=0$.

Рол: $f \in C[a,b] \cap \mathcal{D}(a,b)$, $f(a)=f(b) \Rightarrow \exists c \in (a,b)\ \ f'(c)=0$. $\rightsquigarrow$



Лагранж: $f \in C[a,b] \cap \mathcal{D}(a,b) \Rightarrow \exists c \in (a,b)\ \ f'(c) = \dfrac{f(b)-f(a)}{b-a}$

обе две праве су паралелне

Коши: $f, g \in C[a,b] \cap \mathcal{D}(a,b)$, $\forall x \in (a,b)\ \ g'(x) \neq 0$ $\quad (\Rightarrow g(a) \neq g(b))$

$$\Rightarrow \exists c \in (a,b) \quad \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Последица: $\forall x \in (a,b): f'(x)=0 \Rightarrow f = const$

Дарбу: $f \in C[a,b] \cap \mathcal{D}(a,b)$, $c,d \in (a,b)$ $c<d$

$$\Rightarrow \forall \gamma \in (f'(c), f'(d)) \quad \exists \xi \in (c,d) \quad f'(\xi)=\gamma.$$

(1) $a_0, \dots, a_n \in \mathbb{R}$, $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_n}{n+1} = 0$

$\overset{?}{\Rightarrow}$ $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ има реалну нулу.

$p'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + a_1$ $\rightsquigarrow$ не видимо везу са условом

$g(x) = ?$

$g'(x) = p(x)$

$b_k (x^k)' = b_k k \cdot x^{k-1} = a_{k-1} x^{k-1}$

$\underline{\underline{b_k}} = \dfrac{a_{k-1}}{k}$ $\forall x^k$ $\Rightarrow g(x) = \dfrac{a_n x^{n+1}}{n+1} + \dfrac{a_{n-1}x^n}{n} + \dots + \dfrac{a_1 x^2}{2} + \dfrac{a_0 x}{1} + 0$

$$g'(x) = \frac{a_n}{\cancel{n+1}} \cdot \cancel{(n+1)} \cdot x^n + \dots + \frac{a_1}{\cancel{2}} \cdot \cancel{2}x + a_0 = p(x)$$

$x=1$

$$g(1) = \frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \dots + \frac{a_1}{2} + \frac{a_0}{1} = 0$$

$$g(0) = \frac{a_n}{n+1} 0^{n+1} + \dots + \frac{a_1}{2} \cdot 0^2 + a_0 \cdot 0 = 0$$

$\Big\} \Rightarrow g(0)=g(1)=0$, $g \in C[0,1] \cap \mathcal{D}(0,1)$ $\Big\} \overset{\text{Рол}}{\Rightarrow} \exists c \in (0,1)\ \ g'(c)=0$, $g'(c) = p(c) = 0$

② $f \in C[0,1] \cap \mathcal{D}(0,1) \overset{?}{\Rightarrow} \exists c \in (0,1):\quad f(1)-f(0) = \underbrace{\frac{1}{2}\left(f'(c)+f'(1-c)\right)}$

$F(x) = ?$

$F'(x) = \frac{1}{2}\left(f'(x)+f'(1-x)\right)$

$\exists c \in (0,1):\quad F(1)-F(0) = F'(c) = \frac{1}{2}\left(f'(c)+f'(1-c)\right)$

$\underset{f(1)-f(0)}{\overset{\shortparallel}{}} \rightarrow$ волели смо да ово важи али не мора бити тачно

$F(x) = \frac{1}{2}\left(f(x) - f(1-x)\right) \rightarrow F'(x) = \frac{1}{2}\left(f'(x)+f'(1-x)\right)$

$F(0) = \frac{1}{2}\left(f(0) - f(1-0)\right) = \frac{1}{2}\left(f(0)-f(1)\right)$

$F(1) = \frac{1}{2}\left(f(1) - f(1-1)\right) = \frac{1}{2}\left(f(1)-f(0)\right)$

$F(1)-F(0) = \frac{1}{2}\left(f(1)-f(0) - \left(f(0)-f(1)\right)\right) = f(1)-f(0)$

$F \in C[0,1] \cap \mathcal{D}(0,1) \overset{\text{Лагранж}}{\Longrightarrow} \underset{\exists c \in (0,1)}{}:\ \frac{F(1)-F(0)}{1-0} = F'(c)$

$f(1)-f(0) = \frac{1}{2}\left(f'(c)+f'(1-c)\right).$

③ $f,g \in C[a,b] \cap \mathcal{D}(a,b)\quad f(a)=f(b)=0,\quad g(a)g(b) \neq 0,\quad \forall x \in (a,b)\ f'(x)g(x) \neq f(x)g'(x)$

$\overset{?}{\Rightarrow} \exists c \in (a,b)\quad g(c)=0.$

$g(a) \neq 0$ $\qquad f(a)=f(b)=0$

$g(b) \neq 0$

$\forall x \in (a,b)\quad f'(x)g(x) \neq f(x)\cdot g'(x) \iff f'(x)g(x) - f(x)g'(x) \neq 0$

ПНС. $g(x) \neq 0\quad \forall x \in [a,b]$ $\qquad f\cdot g = f'g + g'f$

$F(x) = \frac{f(x)}{g(x)}$

$F'(x) = \frac{f'(x)g(x) - f(x)\cdot g'(x)}{g^2(x)} \neq 0 \quad \forall x \in (a,b)$

$F \in C[a,b] \cap \mathcal{D}(a,b)$

$F(a) = \frac{f(a)}{g(a)} = 0,\quad F(b) = \frac{f(b)}{g(b)} = 0$

Рол

$\Rightarrow \exists c \in (a,b)\quad F'(c)=0$ ↯ $\qquad \Rightarrow \exists \xi \in (a,b)\quad g(\xi)=0$

(4) $f \in C[a,b] \cap \mathcal{D}(a,b)$, $f(a)=a$, $f(b)=b$

$\overset{?}{\Rightarrow} \exists\, s,t : a<s<t<b \quad \dfrac{1}{f'(s)}+\dfrac{1}{f'(t)}=2$

Лагранж $\exists c\in(a,b)$ $\quad f'(c)=\dfrac{f(b)-f(a)}{b-a}=\dfrac{b-a}{b-a}=1$

$x,y\in(a,b) \overset{\text{Лагранж}}{\Longrightarrow} \exists \xi \in (\overleftrightarrow{x,y}) \quad f'(\xi)=\dfrac{f(x)-f(y)}{x-y} \longrightarrow \dfrac{1}{f'(\xi)}=\dfrac{x-y}{f(x)-f(y)}$

$f\in C[\overleftrightarrow{x,y}]\cap \mathcal{D}(\overleftrightarrow{x,y})$

$f(a)=a$ $\quad$? $x_0\in(a,b)$, $f(x_0)=?$: $\quad b-f(x_0)=f(x_0)-a \to f(x_0)=\dfrac{b+a}{2}\in(a,b)$

$f(b)=b$ $\quad$ Болцано - Коши, $f\in C[a,b] \Rightarrow \exists x_0\in(a,b)$ $f(x_0)=\dfrac{a+b}{2}$

$s\in(a,x_0) \quad \dfrac{1}{f'(s)}=\dfrac{x_0-a}{f(x_0)-a}$

$t\in(x_0,b) \quad \dfrac{1}{f'(t)}=\dfrac{b-x_0}{b-f(x_0)}$

$$\left.\right\} \quad \frac{1}{f'(s)}+\frac{1}{f'(t)}=\frac{b-x_0}{b-f(x_0)}+\frac{x_0-a}{f(x_0)-a}=$$

$$=\frac{b-x_0}{\frac{b-a}{2}}+\frac{x_0-a}{\frac{b-a}{2}}=\frac{b-a}{\frac{b-a}{2}}=2.$$

$a<s<t<b$

(5) $f\in\mathcal{D}(\mathbb{R})$, $a\neq 0$ $\quad f(a)=1 \quad \overset{?}{\Rightarrow} \exists c\in\mathbb{R} \quad \underbrace{c\cdot f'(c)+f(c)}_{(x\cdot f(x))'\big|_{x=c}}=\underset{(x')}{\overset{}{1}}.$

$? \exists c\in\mathbb{R}: \quad \left((x\cdot f(x))'-(x)'\right)\Big|_{x=c}=0$

$\quad\quad (x\cdot f(x)-x)'\Big|_{x=c}=0$

$\leadsto F(x)=x\cdot f(x)-x$, $\exists c: F'(c)=0$? $\quad \longrightarrow F'(x)=x\cdot f'(x)+f(x)-1$

$F(a)=a\cdot f(a)-a=a-a=0$

$F(1)=1\cdot f(1)-1$

$F(0)=0\cdot f(0)-0=0$

$F\in\mathcal{D}(\mathbb{R}) \Rightarrow F\in C[\overleftrightarrow{0,a}]\cap\mathcal{D}(\overleftrightarrow{0,a})$, $F(0)=F(a)=0$

$\overset{\text{Рол}}{\Longrightarrow} \exists c\in(\overleftrightarrow{0,a}) \quad F'(c)=0=c\cdot f'(c)+f(c)-1$

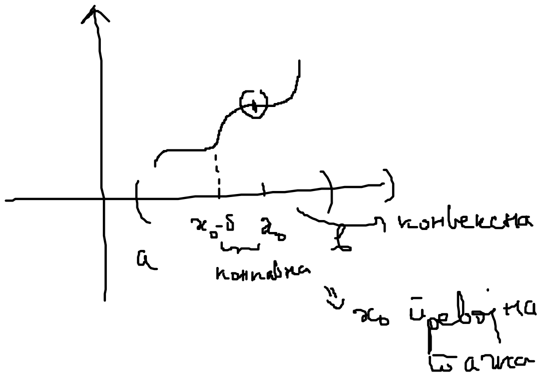
$\Rightarrow c\cdot f'(c)+f(c)=1$

(6) $f\in C[-1,1]\cap\mathcal{D}^2(-1,1)$ има превојну тачку, $f(0)=0$, $f(1)+f(-1)>0$

$\overset{?}{\Rightarrow} \exists c\in(-1,1): \quad \dfrac{f(-1)+f(1)}{2}=f''(c).$

$f \quad \cap \quad \cup$

$f\in C(a,b)$, $x_0\in(a,b)$ је превојна тачка ф-је f ако ф-ја f конвексна (конкавна) у некој левој околини x_0 $(x_0-\delta,x_0)$, и f конкавна (конвексна) у десној околини x_0 $(x_0,x_0+\delta)$.



Ако f двапут диф у x_0 $(\exists f''(x_0))$ и x_0 је превојна $\Rightarrow f''(x_0)=0$ $\not\Leftarrow$

у овој тачки се мења знак другом изводу.

$f''>0$ на $(x_0-\delta, x_0) \Rightarrow f$ конвексна на $(x_0-\delta, x_0)$

$f''<0$ на $(x_0-\delta, x_0) \Rightarrow f$ конкавна на $(x_0-\delta, x_0)$

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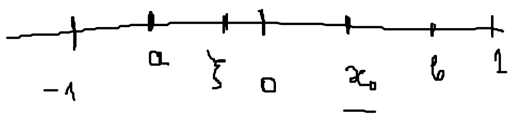
f има превојну тачку $\Rightarrow \exists x_0$ превојна тачка $f''(x_0)=0$

$f \in D^2(a,b)$

$f(1)+f(-1)>0$

$f(0)=0$



Лагранж на $(-1,0)$: $f \in C[-1,0] \cap D(-1,0)$

$\Rightarrow \exists a \in (-1,0): \; f'(a) = \dfrac{f(0)-f(-1)}{0-(-1)} = -f(-1)$

Лагранж на $(0,1)$:

$\Rightarrow \exists b \in (0,1) \quad f'(b) = \dfrac{f(1)-\overset{=0}{f(0)}}{1-0} = f(1)$

$f' \in C[a,b] \cap D(a,b)$ јер $\exists f''$ на $(-1,1)$

$\underset{\Rightarrow}{\text{Лагранж}}$ $\exists \xi \in (a,b) \quad f''(\xi) = \dfrac{f'(b)-f'(a)}{b-a} = \dfrac{f(1)-(-f(-1))}{b-a} = \dfrac{f(1)+f(-1)}{b-a}$

? $\exists c \in (-1,1): \quad f''(c) = \dfrac{f(1)+f(-1)}{2}$?

$0 < b-a < 2 \qquad 0 < \dfrac{f(1)+f(-1)}{2} < \dfrac{f(1)+f(-1)}{b-a}$

$\exists x_0: \quad f''(x_0)=0 \quad , x_0 \neq \xi$

$\Downarrow$

$f' \in C[x_0,\xi] \cap D(x_0,\xi) \rightarrow$ јер $\exists f''$

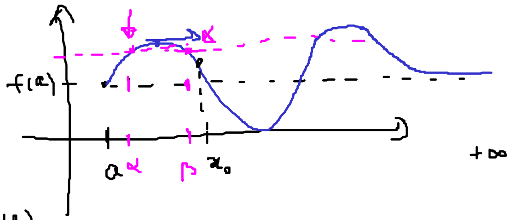
$\underset{\text{Дарбу}}{\Rightarrow}$ $\forall \gamma \in (\underset{0}{\overset{\parallel}{f''(x_0)}}, f''(\xi)) \quad \exists \theta \in (x_0,\xi) \quad f''(\theta)=\gamma$

$0 < \dfrac{f(1)+f(-1)}{2} < f''(\xi)$

$\Rightarrow \exists c \in (x_0,\xi) \quad f''(c) = \dfrac{f(1)+f(-1)}{2}$

(6) $f\in C[a,+\infty)\cap D(a,+\infty)$ $\quad \lim\limits_{x\to+\infty} f(x)=f(a)$

$\Rightarrow \exists b\in(a,+\infty):\ f'(b)=0.$
?



идеја: да покажемо да $\exists \alpha,\beta\in[a,+\infty)$ $\quad f(\alpha)=f(\beta)$

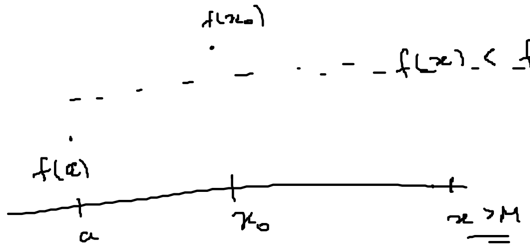
$\overset{\text{Рол}}{\Rightarrow}\ \exists b\in(\alpha,\beta)\quad f'(b)=0.$

1° $f = const \Rightarrow f'(x)=0 \quad \forall x\in(a,+\infty)$

2° $f\neq const \Rightarrow \exists x_0\in(a,+\infty)\quad f(x_0)\neq f(a)$, БУО $f(x_0)>f(a)$

$\lim\limits_{x\to+\infty} f(x)=f(a) \Rightarrow \forall\varepsilon>0\ \exists M>0\quad x>M \quad |f(x)-f(a)|<\varepsilon$

$f(x)\in(f(a)-\varepsilon, f(a)+\varepsilon)$



$f(x) < f(a)+\varepsilon < f(x_0)$

$\Downarrow$

$\varepsilon < f(x_0)-f(a)$

$\varepsilon = \dfrac{f(x_0)-f(a)}{2} > 0$

$\underline{\underline{x>M}} \Rightarrow f(x) < f(a)+\varepsilon = \dfrac{f(x_0)+f(a)}{2}$

$x>\max\{M,x_0\} \Rightarrow f(x) < \dfrac{f(x_0)+f(a)}{2} < f(x_0)$

Болцано-Коши

$f\in C[x_0,x] \Rightarrow \exists \beta\in(x_0,x):\quad f(\beta)=\dfrac{f(x_0)+f(a)}{2}$

$\|$

$f\in C[a,x_0] \Rightarrow \exists \alpha\in(a,x_0)\quad f(\alpha)=\dfrac{f(x_0)+f(a)}{2}$

$f(a)<f(x_0)\qquad \alpha<\beta \quad \overset{\text{Рол}}{\Rightarrow}\ \exists b\in(\alpha,\beta)\quad f'(b)=0.$

$f\in C[\alpha,\beta]\cap D(\alpha,\beta)$

$f(\alpha)=f(\beta)$

$$\lim_{x\to0^+} x^2\ln x = \lim_{x\to0^+}\frac{\ln x}{\frac{1}{x^2}} \overset{\text{ЛП}}{=} \lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x\to0^+}\frac{1}{-\frac{2}{x^2}} = \lim_{x\to0^+} -\frac{x^2}{2} = 0$$

$$e^{x^2\ln x} \xrightarrow[x\to0]{} 1$$

$$a\cdot\lim_{x\to0^-}\frac{-x\sqrt{1+\dots}}{x^b} = 1$$

$\to 1$

$\sim -x^{1-b}$

$$\lim x^{1-b} = \begin{cases} \boxed{1, \quad b=1} \\ 0, \quad b<1 \\ \text{не постоји } +\infty, \quad b>1 \end{cases}$$

1 за $b=1 \implies a=-1$

$$f'_+(0) = \lim_{h\to0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to0^+}\frac{e^{\overset{\to 0}{h^2\ln h}}-1}{h} = \lim_{h\to0^+}\frac{\cancel{1}+h^2\ln h+o(h^2\ln h)-\cancel{1}}{h} =$$

$$= \lim_{h\to0^+} \underbrace{h\cdot\ln h}_{\to0} + \underbrace{o(h\ln h)}_{\to0} = 0$$

$$\lim_{h\to0^+} h\cdot\ln h = \lim_{h\to0^+}\frac{\ln h}{\frac{1}{h}} \overset{\text{ЛП}}{=} \lim_{h\to0^+}\frac{\frac{1}{h}}{-\frac{1}{h^2}} = \lim_{h\to0^+} -h = 0$$

$$\sin h^2 \sim h^2 - \underbrace{\frac{(h^2)^3}{3!}+o(h^6)}_{o(h^4)}, \quad h\to0$$

$$f'_-(0) = \lim_{h\to0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to0^-}\frac{-\frac{\sqrt{\sin h^2-h^3}}{h}-1}{h} = \lim_{h\to0^-}\frac{-\sqrt{\sin h^2-h^3}-h}{h^2} =$$

$\sim 1+\frac{1}{2}(-h+o(h^2))+o(h)$

$$= \lim_{h\to0^-}\frac{-\sqrt{h^2+o(h^4)-h^3}-h}{h^2} = \lim_{h\to0^-}\frac{-(-h)\sqrt{1-h+o(h^2)}-h}{h^2}$$

$$= \lim_{h\to0^-}\frac{h\cdot(\cancel{1}-\frac{1}{2}h+o(h))-\cancel{h}}{h^2} = \lim_{h\to0^-}\frac{-\frac{1}{2}h^2+o(h^2)}{h^2} = -\frac{1}{2} \neq 0 = f'_+(0)$$

$\implies f$ није диф. у 0.