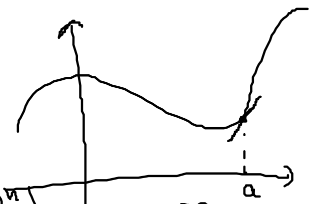
Тејлоров развој ф-ја

$f\in C^n(c,d)$ n-пута диф. ф-ја, $a\in(c,d)$



$$f(x)=f(a)+\frac{f'(a)}{1!}\cdot(x-a)+\frac{f''(a)}{2!}\cdot(x-a)^2+\dots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\underbrace{o((x-a)^n)}_{\text{Пеанов остатак}},\quad x\to a$$

уколико $f\in D^{n+1}((a,b))$

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot(x-a)^{n+1}}_{\text{Лагранжев остатак}},\quad x\to a$$

$\xi\in(x,a)$

$a=0\rightarrow$ Маклоренов развој

$$f(x)=f(0)+\frac{f'(0)}{1!}\cdot x+\frac{f''(0)}{2!}x^2+\dots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n),\quad x\to 0$$

$$\frac{f^{(n+1)}(\xi)\cdot x^{n+1}}{(n+1)!},\quad x\to 0$$

- $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+o(x^n),\quad x\to 0$

$f(x)=e^x\Rightarrow f'(x)=e^x,\ f''(x)=e^x,\ \dots,\ f^{(n)}(x)=e^x$

$f(0)=f'(0)=\dots=f^{(n)}(0)=e^0=1$

- $\sin x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+o(x^{2n+2}),\quad x\to 0$

$f(x)=\sin x,\quad f'(x)=\cos x,\quad f''(x)=-\sin x$

$f'''(x)=-\cos x,\quad f^{(4)}(x)=\sin x$

- $\cos x=1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots+(-1)^n\frac{x^{2n}}{(2n)!}+o(x^{2n+1}),\quad x\to 0$

- $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots+(-1)^{n-1}\frac{x^n}{n}+o(x^n),\quad x\to 0$

$f'(x)=\frac{1}{1+x},\quad f''(x)=((1+x)^{-1})'=-(1+x)^{-2}$

$f'''(x)=2\cdot(1+x)^{-3}$

- $(1+x)^\alpha=1+\binom{\alpha}{1}x+\binom{\alpha}{2}x^2+\dots+\binom{\alpha}{n}x^n+o(x^n),\quad x\to 0$

$$\binom{\alpha}{k}=\frac{\alpha\cdot(\alpha-1)\dots(\alpha-k+1)}{k!},\quad \alpha\in\mathbb{R},\ k\in\mathbb{N}$$

① Развој ф-је f у околини тачке a до степена n.

а) $f(x) = \frac{1}{1+x}$, $a=0$, $n \in \mathbb{N}$

$\alpha = -1$: $f(x) = (1+x)^{-1} = 1 + \binom{-1}{1}x + \binom{-1}{2}x^2 + \dots + \binom{-1}{n}x^n + o(x^n)$, $x \to 0$

$$\binom{-1}{k} = \frac{-1\cdot(-1-1)\cdot(-1-2)\cdots(-1-k+1)}{k!} = \frac{-1\cdot(-2)\cdot(-3)\cdots(-k)}{k!} = \frac{(-1)^k\cdot k!}{k!} = (-1)^k$$

$$f(x) = 1 - x + x^2 - x^3 + x^4 + \dots + (-1)^n x^n + o(x^n), \ x \to 0$$

или

$$(\ln(1+x))' = \left(x - \frac{x^2}{2} + \dots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)\right)', \ x \to 0$$

$$\frac{1}{1+x} = 1 - x + \dots + (-1)^{n-1}x^{n-1} + o(x^{n-1}).$$

б) $f(x) = \operatorname{tg} x$, $a = 0$, $n = 4$

$f'(x) = \frac{1}{\cos^2 x} = (\cos x)^{-2}$

$f''(x) = -2(\cos x)^{-3}\cdot(-\sin x) = 2(\cos x)^{-3}\cdot\sin x$

$f'''(x) = -6\cdot(\cos x)^{-4}\cdot(-\sin^2 x) + 2(\cos x)^{-2}$

$f^{(4)}(x) = \dots$

$$f(x) = \frac{\sin x}{\cos x} = \sin x\cdot(\cos x)^{-1}$$

$\sin x = x - \frac{x^3}{3!} + o(x^4)$, $x \to 0$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)$, $x \to 0$

$$(\cos x)^{-1} = (\underbrace{1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)}_{t})^{-1} = 1 - \left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)\right) + \left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)\right)^2 + o\left(\left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)\right)^2\right)$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{4!} + o(x^5) + x^4\left(-\frac{1}{2} + \frac{x^2}{4!} + o(x^3)\right)^2 + o(x^4)$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^4}{4} + o(x^4), \ x \to 0$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + o(x^4), \ x \to 0$$

$$\operatorname{tg} x = \sin x \cdot (\cos x)^{-1} = \left(x - \frac{x^3}{6} + o(x^4)\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + o(x^4)\right)$$

$$= x + \frac{x^3}{2} + \underbrace{\frac{5x^5}{24} + o(x^5)} - \frac{x^3}{6} + o(x^4) = x + \frac{x^3}{3} + o(x^4), \ x \to 0$$

в) $\operatorname{arctg} x$, $n=2$, $a=5$

$$\operatorname{arctg} x = \operatorname{arctg} 5 + \frac{x-5}{26} - \frac{\overset{5}{\cancel{10}}}{(26)^2}\cdot\frac{(x-5)^2}{8} + o((x-5)^2), \quad x\to 5$$

$x^2-10x+25$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$f''(x) = -(1+x^2)^{-2}\cdot 2x \qquad = \dots$$

$$f'''(x) = 2(1+x^2)^{-3}\cdot 4x - 2\cdot(1+x^2)^{-2} \qquad f(x) = f(a) + f'(a)\cdot\frac{x-a}{1!}$$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + o(x^3), \quad x\to 0$$

② $$\lim_{n\to\infty}\left(e^{\sin\frac{1}{n}}\cos\left(\sin\frac{1}{n}\right) - 1 - \frac{1}{n}\right)n^3$$

$$\sin x = x - \frac{x^3}{6} + o(x^4), \quad x\to 0$$

$x=\frac{1}{n}$ $$\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{6n^3} + o\left(\frac{1}{n^4}\right), \quad n\to\infty$$

$\sim\frac{1}{n}$

$$e^{\sin\frac{1}{n}} = 1 + \frac{\sin\frac{1}{n}}{1!} + \frac{(\sin\frac{1}{n})^2}{2!} + \frac{(\sin\frac{1}{n})^3}{3!} + o\left((\sin\tfrac{1}{n})^3\right)$$

$x=\sin\frac{1}{n}$

$$= 1 + \frac{\frac{1}{n} - \frac{1}{6n^3} + \underbrace{o\left(\frac{1}{n^4}\right)}_{=o\left(\frac{1}{n^3}\right)}}{1!} + \frac{\left(\frac{1}{n} - \frac{1}{6n^3} + o\left(\frac{1}{n^4}\right)\right)^2}{2} + \frac{\left(\frac{1}{n} - \frac{1}{6n^3} + o\left(\frac{1}{n^4}\right)\right)^3}{6} + o\left(\frac{1}{n^3}\right)$$

$$= 1 + \frac{1}{n} - \cancel{\frac{1}{6n^3}} + \frac{1}{2n^2} + o\left(\frac{1}{n^3}\right) + \cancel{\frac{1}{6n^3}} + o\left(\frac{1}{n^3}\right)$$

$$= 1 + \frac{1}{n} + \frac{1}{2n^2} + o\left(\frac{1}{n^3}\right), \quad n\to+\infty$$

$$\cos\left(\sin\frac{1}{n}\right) = 1 - \frac{(\sin\frac{1}{n})^2}{2} + o\left((\sin\tfrac{1}{n})^3\right) =$$

$\sim\frac{1}{n}$

$\ulcorner$ $f\sim g$ $\quad o(f) = o(g)$

$f = g + o(g)$

$(\sin\frac{1}{n})^3 = \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right)^3 = \frac{1}{n^3}$

$o(\ \ ) = o\left(\frac{1}{n^3}\right)$

$$= 1 - \frac{\left(\frac{1}{n} - \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)\right)^2}{2} + o\left(\frac{1}{n^3}\right)$$

$$= 1 - \left(\frac{\frac{1}{n^2} - \frac{1}{3n^4} + \frac{1}{36n^6} + o\left(\frac{1}{n^6}\right) + o\left(\frac{1}{n^4}\right) + o\left(\frac{1}{n^6}\right)}{2}\right) + o\left(\frac{1}{n^3}\right)$$

$$= 1 - \frac{1}{2n^2} + o\left(\frac{1}{n^3}\right), \quad n\to+\infty$$

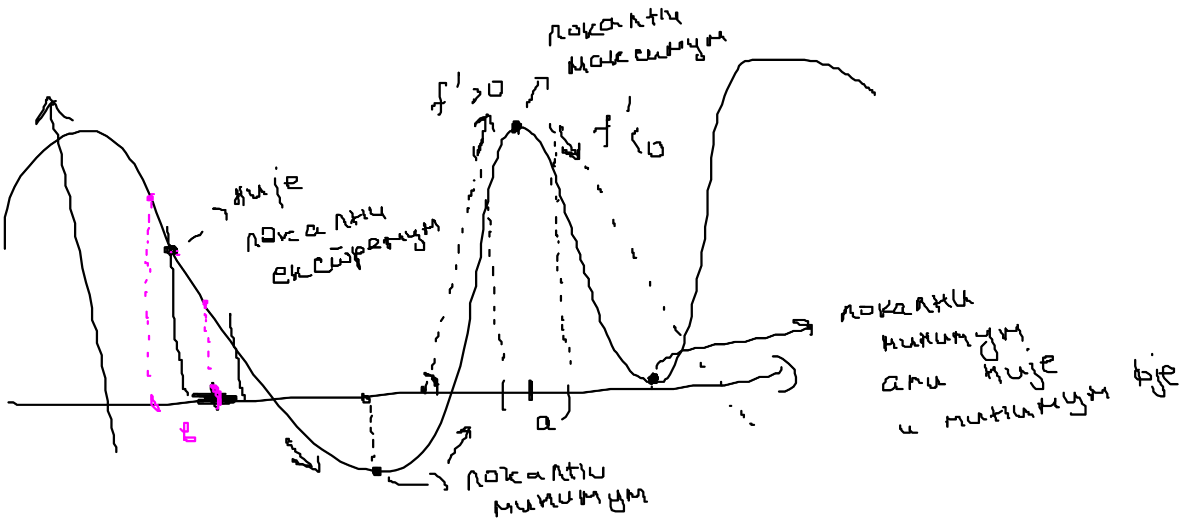
$$\lim_{n\to\infty}\left(e^{\sin\frac{1}{n}}\cos\left(\sin\tfrac{1}{n}\right) - 1 - \frac{1}{n}\right)n^3 = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}+\frac{1}{2n^2}+o\left(\frac{1}{n^3}\right)\right)\cdot\left(1-\frac{1}{2n^2}+o\left(\frac{1}{n^3}\right)\right) - 1 - \frac{1}{n}\right)n^3$$

$$= \lim_{n\to\infty}\left(\cancel{1} + \cancel{\frac{1}{n}} + \cancel{\frac{1}{2n^2}} + o\left(\frac{1}{n^3}\right) - \cancel{\frac{1}{2n^2}} - \frac{1}{2n^3} - \frac{1}{4n^4} + o\left(\frac{1}{n^3}\right) - \cancel{1} - \cancel{\frac{1}{n}}\right)\cdot n^3 =$$

$$f\sim g \Rightarrow o(f^n) = o(g^n)$$

$$= \lim_{n\to\infty}\left(-\frac{1}{2n^3}+o\left(\frac{1}{n^3}\right)\right)\cdot n^3 = \lim_{n\to\infty} -\frac{1}{2}+o(1) = -\frac{1}{2}$$

③ $\lim_{x\to+\infty}\left(\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}\right)$ за вежбу



Фермаова Т. $f: A\to\mathbb{R}$, $a\in A$ и f диф a

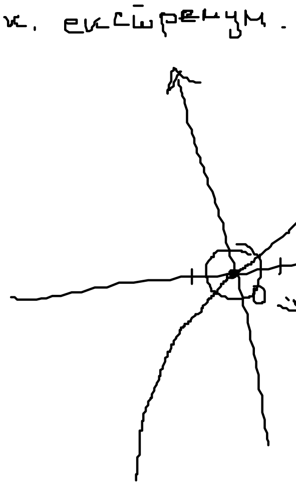
f има локални екстремум у a $\Rightarrow$ $f'(a)=0$.

a је локални минимум ако $\exists\,\delta>0 : \forall x\in(a-\delta,a+\delta)$ $f(x)\geq f(a)$

a је локални максимум ако $\exists\,\delta>0$ $\forall x\in(a-\delta,a+\delta)$ $f(x)\leq f(a)$

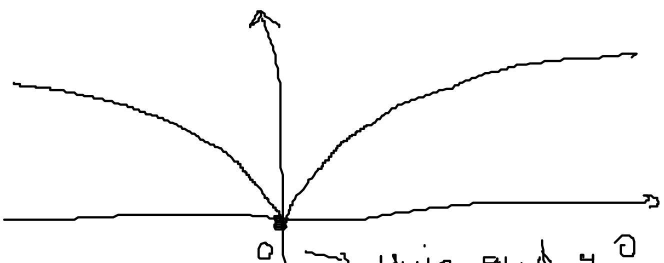
$f(x)=x^2 \Rightarrow f'(x)=2x$, $f'(0)=0 \Rightarrow 0$ лок. екстремум.

$f(x)=x^3 \Rightarrow f'(x)=3x^2$ $f'(0)=0$



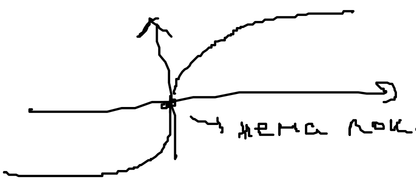
$\Rightarrow$ није лок. екстремум иако $f'(0)=0$.

$$f(x)=\sqrt{|x|} \Rightarrow f'(x)=\begin{cases}\frac{1}{2\sqrt{x}}, & x>0\\ \text{није деф.} & x=0\\ -\frac{1}{2\sqrt{|x|}}, & x<0\end{cases}$$



→ није диф у 0 али је 0 лок. екстремум

$f(x)=\sqrt[3]{x} \Rightarrow f'(x)$ није деф у 0



→ нема лок. екстремум у 0.