T. [Вајерштрас]: $f:[a,b]\to\mathbb{R}$ непр $\Rightarrow$ f огран и $\exists \min\limits_{[a,b]} f(x) = f(x_1)$

$\exists \max\limits_{[a,b]} f(x) = f(x_2)$

$x_1, x_2 \in [a,b]$.

T. [Болцано - Коши]: $f:[a,b]\to\mathbb{R}$ непр и $f(a)=A$, $f(b)=B$

$\Rightarrow \forall C \in \overset{\leftrightarrow}{[A,B]}$ $\exists c\in[a,b]$ $f(c)=C$.

① $f, g : [a,b]\to[c,d]$ непр., g НА

$\overset{?}{\Rightarrow} \exists x_0\in[a,b]$ $f(x_0)=g(x_0)$.

— — — —

$F(x) = f(x)-g(x)$ непрекидна $[a,b]$

$\exists x_0\in[a,b]$ $F(x_0)=0$.

(*) Да ли $\exists \alpha,\beta\in[a,b]$ $F(\alpha)<0$ и $F(\beta)>0$?

Ако постоје $\Rightarrow$ F је непр на $\overset{\leftrightarrow}{[\alpha,\beta]}$

$\overset{\text{Болцано-Коши}}{\Longrightarrow}$ $\forall C\in[\underset{<0}{F(\alpha)}, \underset{>0}{F(\beta)}]$ $\exists\gamma\in\overset{\leftrightarrow}{[\alpha,\beta]}$ $F(\gamma)=C$

$C=0$ $\exists\gamma\in\overset{\leftrightarrow}{[\alpha,\beta]}$ $F(\gamma)=0$

$\Rightarrow x_0=\gamma$

$\alpha,\beta\in[a,b] \Rightarrow \gamma\in[a,b] \Rightarrow x_0\in[a,b]$, $F(x_0)=0$.

g НА

$\Downarrow$

$\max\limits_{x\in[a,b]} g(x) = d$

$\exists\alpha\in[a,b]$ $g(\alpha)=d$

$F(\alpha)=f(\alpha)-g(\alpha)=f(\alpha)-d\le 0$

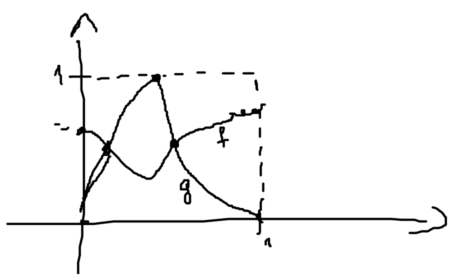
$F(\alpha)=0 \Rightarrow x_0=\alpha$.

$\exists\beta\in[a,b]$ $g(\beta)=c$

$F(\beta)=f(\beta)-g(\beta)=f(\beta)-c\ge 0$

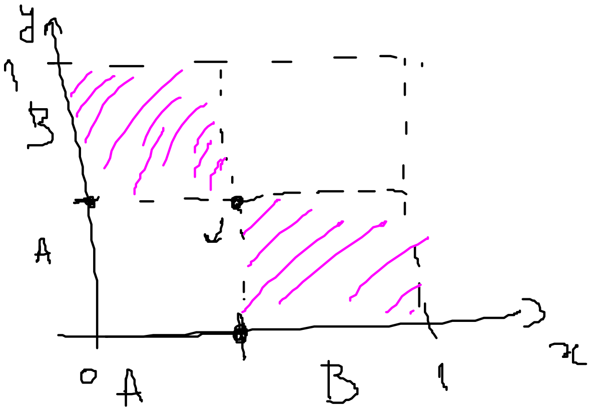
$F(\beta)=0 \Rightarrow x_0=\beta$

За $F(\alpha)<0$ и $F(\beta)>0$ $\Rightarrow$ (*) нам даје x_0.



② Да ли $\exists\ A, B \subseteq [0,1]$, $A \cap B = \emptyset$, $A \cup B = [0,1]$

тако да $\exists\ f : [0,1] \to [0,1]$ непр и $f(A) \subseteq B$ и $f(B) \subseteq A$?



$g : [0,1] \to [0,1]$ НА

$g(x) = x$

$\Downarrow$ ①

$\exists\ x_0 \in [0,1]$ $f(x_0) = g(x_0) = x_0$

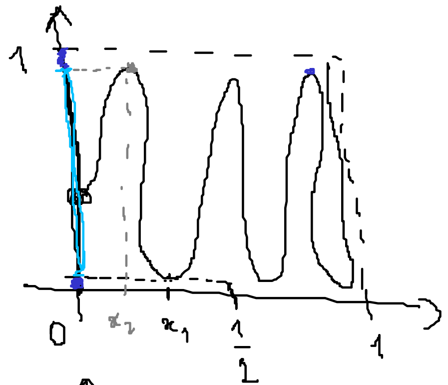
$x_0 \in A$ $\quad \underset{\underset{x_0}{\parallel}}{f(x_0)} \in f(A) \subseteq B \Rightarrow x_0 \in B$ ↯ $A \cap B = \emptyset$

$x_0 \in B$ $\quad f(x_0) \in f(B) \subseteq A \Rightarrow x_0 \in A$ ↯ $A \cap B = \emptyset$

$\Rightarrow$ Овакви A и B не постоје.

③ $f : [0,1) \to (0,1)$ непр и НА

$\overset{?}{\Rightarrow} f\restriction_{(\frac{1}{2},1)} : (\frac{1}{2}, 1) \to (0,1)$ је НА.

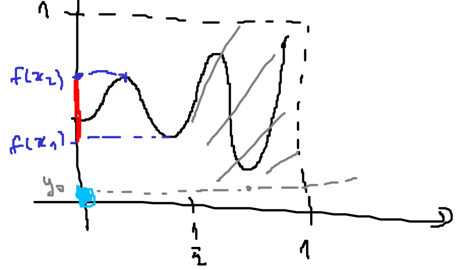


$[0,1) \setminus (\frac{1}{2}, 1) = [0, \frac{1}{2}]$

f је непр на $[0, \frac{1}{2}] \ni x_1$

Вајерштрас $\Rightarrow$ $\exists \min\limits_{[0,\frac{1}{2}]} f(x) = f(x_1) \in (0,1) \Rightarrow \underline{f(x) \geq f(x_1)}$, $x \in [0, \frac{1}{2}]$ ($f(x_1) > 0$)

$(0, f(x_1)) \cap f([0, \frac{1}{2}]) = \emptyset$



$\exists \max\limits_{[0,\frac{1}{2}]} f(x) = f(x_2) \in (0,1)$

$f(x_2) < 1$

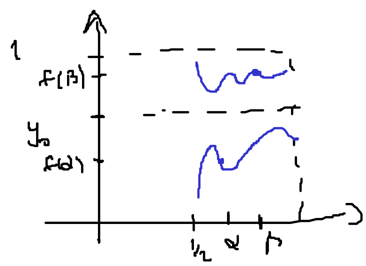
$x_2 \in [0, \frac{1}{2}] \Rightarrow f(x) \leq f(x_2)$, $x \in [0, \frac{1}{2}]$

ппс.

$\exists\ y_0 \in (0,1)$ $\quad \forall x \in (\frac{1}{2}, 1)$ $\quad f(x) \neq y_0$ $\quad$ ($f\restriction_{(\frac{1}{2},1)}$ није НА)

$x \in (\frac{1}{2}, 1)$ $\quad f(x) \in$?

? $\exists\ \alpha \in (\frac{1}{2}, 1)$, $\beta \in (\frac{1}{2}, 1)$ $\quad f(\alpha) < y_0 < f(\beta)$?



Ако ∃овакви α и β

$f\restriction_{\overleftrightarrow{(\alpha,\beta)}} : \overleftrightarrow{(\alpha,\beta)} \to (0,1)$ нейр.

Болцано-Коши : $\exists x \in \overleftrightarrow{(\alpha,\beta)}$ $f(x) = y_0$ ↯

$\Rightarrow$ $f(x) > y_0$ $\forall x \in (\frac{1}{2},1)$ или $f(x) < y_0$ $x \in (\frac{1}{2},1)$

БУО $f(x) > y_0$ $\forall x \in (\frac{1}{2},1)$

$f((\frac{1}{2},1)) \subseteq (y_0,1)$

$f([0,1)) = f([0,\frac{1}{2}] \cup (\frac{1}{2},1)) = f[0,\frac{1}{2}] \cup f(\frac{1}{2},1) \subseteq [\underset{\vee\atop 0}{f(x_1)}, f(x_2)] \cup \underset{\vee\atop 0}{(y_0},1)$

$\subseteq [\underbrace{\min\{y_0, f(x_1)\}}_{\vee\atop 0}, 1)$

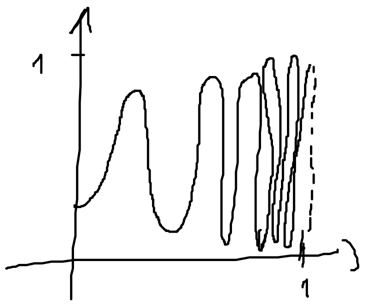
$A \subseteq B$

$A \cap B^c = \emptyset$

$f([0,1)) \cap (0, \underset{\vee\atop 0}{\min\{y_0, f(x_1)\}}) = \emptyset$

f није НА на $(0,1)$ ↯ $\Rightarrow$ $\forall y_0 \in (0,1)$ $\exists x \in (\frac{1}{2},1)$ $f(x) = y_0$.

$f\restriction_{(\frac{1}{2},1)}$ је НА.



$f(x) = \sin\frac{1}{1-x}$ има особину из зад. 3

4 $f: [0,2] \to \mathbb{R}$ нейр.

$f(x) = f(x^2-2x+2)$ $\forall x \in [0,2]$

$\overset{?}{\Rightarrow}$ $f = \text{const}$

- - - -

$f(x) = f(\underbrace{x^2-2x+2}_{x_1}) = f(\underbrace{x_1^2-2x_1+2}_{x_2}) = f(x_2^2-2x_2+2) = \dots$

x_n: $x_0 = x \in [0,2]$

$x_{n+1} = x_n^2 - 2x_n + 2 = (x_n-1)^2 + 1 \geq 1$

(БИ) $x_0 \in [0,2]$ (ИК) $x_n \in [0,2] \Rightarrow x_n - 1 \in [-1,1] \Rightarrow (x_n-1)^2 \in [0,1]$

$\Rightarrow x_{n+1} \in [1,2]$

$f(x_n)$ је добро дефинисано

$f(x_n) = f(x_0) = f(x)$

$\lim\limits_{n\to\infty} f(x_n) = f(x)$

f непр. $\Rightarrow$ ако $\exists \lim\limits_{n\to\infty} x_n = a \Rightarrow \exists \lim\limits_{n\to\infty} f(x_n) = f(a) = f(x)$

$x_n \in [1,2]$ $n\in\mathbb{N}$ ограничен

x_n монотон?

$x_{n+1} \square x_n$

$x_n^2 - 2x_n + 2 \square x_n$

$x_n^2 - 3x_n + 2 \square 0$

$(x_n - 1)(x_n - 2) \square 0$

$\square =$ "$=$" ако $x_n = 1$ или $x_n = 2$

$\square =$ "$<$" ако $x_n \in (1,2)$

$x_1 = 1 \Rightarrow x_2 = x_1 = 1 \Rightarrow \dots x_n = 1 \Rightarrow x_n \to 1$ $a=1$

$x_1 = 2 \Rightarrow x_2 = \dots = 2 \Rightarrow \dots x_n = 2 \Rightarrow x_n \to 2$ $a=2$

$x_1 \in (1,2) \Rightarrow x_2 < x_1 < 2,$ $x_2 = \underbrace{(x_1-1)^2}_{>0} + 1 > 1 \Rightarrow x_2 \in (1,2)$

ПМИ: $x_n \in (1,2)$ $\forall n\in\mathbb{N}$

$x_n \downarrow$

$\Rightarrow x_n$ конвергира $a = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} x_n^2 - 2x_n + 2 = a^2 - 2a + 2$

$a^2 - 3a + 2 = 0$

$(a-1)(a-2) = 0 \Rightarrow \boxed{a=1}$ или $a = 2$.

ово не може јер $x_n \downarrow$ и $x_1 < 2$.

$\lim\limits_{n\to\infty} x_n = 1$

$\Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(1) = f(x)$.

када $x_1 \in (1,2)$?

$x_1 = x^2 - 2x + 2 = (x-1)^2 + 1 \in (1,2) \Leftrightarrow \underline{x \in (0,1)\cup(1,2)}$

$\Rightarrow x \in (0,1) \cup (1,2)$ $f(x) = f(1)$

f непр на $[0,2] \Rightarrow f(0) = \lim\limits_{x\to 0^+} \underbrace{f(x)}_{= f(1)} = f(1)$

$f(2) = f(1)$

$f(1) = f(1)$.

$\Rightarrow f(x) = f(1)$ $\forall x \in [0,2]$

$= \text{const}$.

Равномерна непрекидност

$f: A \to \mathbb{R}$ равн непр на A

$$\forall \varepsilon>0 \quad \exists \underset{\delta(\varepsilon)}{\delta>0} \quad \forall x,y\in A \quad |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$$

f непр на A

$$\forall x\in A\ \forall \varepsilon>0 \quad \exists \underset{\delta(x,\varepsilon)}{\underset{\Downarrow}{\delta>0}} \quad \forall y\in A \quad |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$$

равн непр $\Rightarrow$ непр
$\not\Leftarrow$

Кантор: $A=[a,b]$ f непр $\Rightarrow$ f равн непр на $[a,b]$.

$f: A\to\mathbb{R}$ Липшицова : $\exists L>0 \quad \forall x,y\in A \quad |f(x)-f(y)|\le L|x-y|$.

f Липшицова $\Rightarrow$ f равн непр.

(1) f равн непр $(a,b]$ и f равн непр на $[b,c)$ $\Rightarrow$ f равн непр на (a,c)

—"— A —"— B, $A\cap B\neq\emptyset$ $\Rightarrow$ —"— $A\cup B$

- - -

f равн непр $(a,b]$

$$\forall\varepsilon>0 \quad \exists\delta_1(\varepsilon)>0 \quad \forall x,y\in(a,b] \quad |x-y|<\delta_1(\varepsilon) \Rightarrow |f(x)-f(y)|<\varepsilon$$

f равн непр $[b,c)$

$$\forall\varepsilon>0 \quad \exists\delta_2(\varepsilon)>0 \quad \forall x,y\in[b,c) \quad |x-y|<\delta_2(\varepsilon) \Rightarrow |f(x)-f(y)|<\varepsilon.$$

Хоћемо

$$\forall\varepsilon>0 \quad \exists\delta(\varepsilon)>0 \quad \forall x,y\in(a,c) \quad |x-y|<\delta(\varepsilon) \Rightarrow |f(x)-f(y)|<\varepsilon.$$

$\varepsilon>0$ произвољно, $\delta(\varepsilon)=?$

$x,y\in(a,c)$ произвољни $x<y$

1° $x,y\in(a,b]$ $\quad |x-y|<\delta_1(\varepsilon) \Rightarrow |f(x)-f(y)|<\varepsilon$

2° $x,y\in[b,c)$ $\quad |x-y|<\delta_2(\varepsilon) \Rightarrow$ —"—

3° $x\in(a,b]$, $y\in[b,c)$ $\quad |x-y|<\ ?$ $\quad |f(x)-f(y)|<\varepsilon$

$x \quad b \quad y$ $\quad |x-b|<\delta_1(\varepsilon/2)$ $\quad |f(x)-f(b)|<\varepsilon/2$

$|y-b|<\delta_2(\varepsilon/2)$ $\quad |f(y)-f(b)|<\varepsilon/2$

$$|x-y| = |x-b|+|b-y| < \delta_1(\varepsilon/2)+\delta_2(\varepsilon/2) \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

$$\delta(\varepsilon)=\min\{\delta_1(\varepsilon),\delta_2(\varepsilon),\delta_1(\varepsilon/2)+\delta_2(\varepsilon/2)\}>0.$$