Асимптотске релације

$a\in\overline{\mathbb{R}}$ тачка нагомилавања $A\subseteq\mathbb{R}$ и $f, g: A\to\mathbb{R}$ ф-је.

Ф-ја f је бесконачно мала у односу на g када $x\to a$ ако

$\exists\,\underbrace{U(a,\delta)\setminus\{a\}}_{\mathring{U}(a,\delta)}$ и ф-ја α у овој околини тд $\lim\limits_{x\to a}\alpha(x)=0$ и $f(x)=\alpha(x)\cdot g(x)$, $x\in\mathring{U}(a,\delta)$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\alpha(x)=0 \quad \text{уколико } g(x)\neq 0 \text{ у } \mathring{U}(a,\delta).$$

$\underline{\underline{f=o(g)}}$, $x\to a$; $\exists\alpha:\mathring{U}(a,\delta)$ $\quad f(x)=\alpha(x)\cdot g(x)$, $\lim\limits_{x\to a}\alpha(x)=0$.

$o(g)=\alpha(x)\cdot g(x)$

1° $\underbrace{o(g)}_{f_1}-\underbrace{o(g)}_{f_2}=o(g)$, $x\to a$

(*) $\exists\alpha_1,\alpha_2:\mathring{U}(a,\delta)$ $\quad f_1(x)=\alpha_1(x)\cdot g(x)$ $\quad \lim\limits_{x\to a}\alpha_1(x)=\lim\limits_{x\to a}\alpha_2(x)=0$

$f_2(x)=\alpha_2(x)\cdot g(x)$ $\quad \lim\limits_{x\to a}\alpha_1(x)-\alpha_2(x)=0$.

$$\alpha_1(x)\cdot g(x)-\alpha_2(x)\cdot g(x)=\big(\alpha_1(x)-\alpha_2(x)\big)\cdot g(x)=o(g)$$

2° $\underbrace{o(g)}_{f_1}+\underbrace{o(g)}_{f_2}=o(g)$, $x\to a$

(*) $\Rightarrow$ $\lim\limits_{x\to a}\alpha_1(x)+\alpha_2(x)=0$ $\quad\Rightarrow\quad f_1+f_2=\alpha_1 g+\alpha_2 g=(\alpha_1+\alpha_2)\,g=o(g)$, $x\to a$

3° $\underbrace{o(f)}_{h_1}\,\underbrace{o(g)}_{h_2}=o(f\cdot g)$, $x\to a$

$\exists\alpha_1:\mathring{U}(a,\delta_1)\to\mathbb{R}$ $\quad h_1(x)=\alpha_1(x)\cdot f(x)$, $x\in\mathring{U}(a,\delta_1)$

$\exists\alpha_2:\mathring{U}(a,\delta_2)\to\mathbb{R}$ $\quad h_2(x)=\alpha_2(x)\cdot g(x)$, $x\in\mathring{U}(a,\delta_2)$

$\lim\limits_{x\to a}\alpha_1(x)=\lim\limits_{x\to a}\alpha_2(x)=0$

$h_1(x)\cdot h_2(x)=\alpha_1(x)\cdot f(x)\cdot\alpha_2(x)\cdot g(x)=$ ☺ $\quad x\in\underbrace{\mathring{U}(a,\delta_1)\cap\mathring{U}(a,\delta_2)}_{\text{околина тачке } a}$

☺ $=\big(\alpha_1(x)\,\alpha_2(x)\big)\cdot f(x)\,g(x)$

$\lim\limits_{x\to a}\alpha_1(x)\cdot\alpha_2(x)=0$

4° $f\cdot\underbrace{o(g)}_{h}=o(fg)$, $x\to a$

$\exists\alpha$ $\quad\lim\limits_{x\to a}\alpha(x)=0$ $\quad h=\alpha\cdot g$

$f\cdot h=\alpha\cdot f\cdot g=o(fg)$.

5° $\underbrace{o(o(f))}_{g}=o(f)$, $x\to a$

$\exists \alpha \quad \lim_{x\to a} \alpha(x) = 0 \quad$ и $\quad g(x) = \alpha(x)\cdot \underbrace{o(f)}_{h}$

$\exists \beta \quad \lim_{x\to a} \beta(x) = 0 \quad$ и $\quad h(x) = \beta(x)\cdot f(x)$

$\Rightarrow \quad g(x) = \underbrace{\alpha(x)\cdot\beta(x)}_{\downarrow x\to a \atop 0}\cdot f(x) = o(f)$

$6^\circ \quad \underbrace{o(\overset{=0}{0\cdot f})}_{h} = o(f) \quad , \quad x\to a$

$h(x) = \alpha(x)\cdot 0\cdot f = 0$

$7^\circ \quad \underbrace{o(1)}_{h} \longrightarrow 0 \quad , \quad x\to a$

$\exists \alpha \quad h(x) = \alpha(x)\cdot 1 \quad , \quad \lim_{x\to a}\alpha(x) = 0 = \lim_{x\to a} h(x).$

- $f, g : A \to \mathbb{R}$

f је еквив. g када $x\to a$: $\exists$ фја γ у околини тачке a тд $\quad f(x) = \gamma(x)\, g(x)$

$f\sim g$ $\qquad$ и $\lim_{x\to a}\gamma(x) = 1.$

$\lim_{x\to a}\dfrac{f(x)}{g(x)} = 1 \iff f\sim g \,,\ x\to a$

Тврд: $f\sim g \quad , x\to a \iff f(x) = g(x) + o(g)$

$f\sim g \iff f = \gamma\cdot g \quad , \lim_{x\to a}\gamma(x) = 1 \Rightarrow \lim_{x\to a}\underbrace{\gamma(x)-1}_{\alpha(x)} = 0 \quad \Big\} \Rightarrow \gamma(x) = 1+\alpha(x)$
$\qquad \lim_{x\to a}\alpha(x) = 0$

$f = g + \alpha\cdot g \quad , \quad \alpha\to 0 \,,\ x\to a$

$f = g + o(g) \,,\ x\to a.$

Тврд: $f\sim g \Rightarrow o(f) = o(g) \quad , \quad x\to a$

$h = o(f) \Rightarrow h = \alpha\cdot f \quad , \alpha\to 0 \quad , x\to a$

$f = \gamma\cdot g \quad , \gamma\to 1 \,, x\to a$

$\Rightarrow h = \alpha\cdot\gamma\cdot g \quad , \quad \alpha\cdot\gamma\to 0 \,, x\to a$

$h = o(g).$

$x^n \,,\ x^m \quad , \quad n>m \quad , \quad \underline{\underline{x\to 0}}$

$o(x^n) \qquad o(x^m)$

$h = o(x^n), x\to 0 \overset{?}{\Longrightarrow} \quad h = o(x^m) \,, x\to 0$

$h(x) = \alpha(x)\, x^n \,; \quad \alpha\to 0 \,, x\to 0$

$m < n \Rightarrow x^n = \underbrace{x^{n-m}}_{\downarrow x\to 0 \atop 0} \cdot x^m$

$$h(x) = \Big(\underbrace{\alpha(x)\cdot x^{n-m}}_{\downarrow x\to 0 \atop 0}\Big)\cdot x^m \Rightarrow h = o(x^m), \quad x\to 0$$

$h = o(x^m)$ $\quad \underline{h} = (\alpha\cdot x^{m-n})\cdot x^n \not\Rightarrow h = o(x^n), \quad x\to 0$

$h = x^n = o(x^m), \quad x\to 0$

$x^n \neq o(x^n), \quad x\to 0$

- $n > m, \quad x\to +\infty$

$h = o(x^n) \rightsquigarrow h = \alpha\cdot x^n, \quad \lim\limits_{x\to+\infty}\alpha(x) = 0, \quad h(x) = \underbrace{\alpha(x)\cdot x^{n-m}}_{\text{не знамо шта се дешава}}\cdot x^m$

$\cancel{\Rightarrow}$? $h = o(x^m)$

$h(x) = x^m, \quad h(x) = \underbrace{x^{\overset{<0}{(m-n)}}}_{\downarrow \atop 0}\cdot x^n$

$x^m \neq o(x^m)$

$h = o(x^m), \quad x\to+\infty \Rightarrow h = o(x^n), \quad x\to+\infty$

$$h(x) = \alpha(x)\cdot x^m = \underbrace{\alpha(x)\cdot x^{m-n}}_{\downarrow x\to+\infty \atop 0}\cdot x^n = o(x^n)$$

$$\boxed{\begin{array}{llll} \cdot\ x\to 0 & n>m & h=o(x^n) & \Rightarrow h = o(x^m) \\ \cdot\ x\to+\infty & n>m & h=o(x^m) & \Rightarrow h=o(x^n). \end{array}}$$

нпр.

$x^2, x \Rightarrow$ "$o(x^2) \subseteq o(x)$", $x\to 0$

$h = o(x^2) \Rightarrow h = o(x), \quad x\to 0$

$x^2, x \Rightarrow$ "$o(x) \subseteq o(x^2)$", $x\to+\infty$

$h = o(x) \Rightarrow h = o(x^2)$

$$\boxed{\begin{array}{l} n>m: \\ \underset{h_1}{o(x^n)} + \underset{h_2}{o(x^m)} = o(x^m) \quad x\to 0, \end{array}}$$

$\left(o(x^2) + o(x) = o(x), \quad x\to 0\right)$

$h_1 = \alpha_1\cdot x^n, \quad \alpha_1 \xrightarrow[x\to 0]{} 0$

$h_2 = \alpha_2\cdot x^m \quad \alpha_2 \xrightarrow[x\to 0]{} 0$

$$h_1 + h_2 = \alpha_1 x^n + \alpha_2 x^m = \Big(\underbrace{\alpha_1 x^{n-m} + \alpha_2}_{\downarrow x\to 0 \atop 0}\Big)x^m$$

$$\boxed{o(x^n) + o(x^m) = o(x^n) \quad x\to+\infty}$$

$o(c\cdot f) = o(f), \quad c = \text{const.}$

1° $\lim_{x\to 0}\frac{\sin x}{x}=1 \Rightarrow \sin x \sim x,\ x\to 0 \Rightarrow \sin x = x + o(x),\ x\to 0$

2° $\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e \Rightarrow \lim_{x\to 0}\frac{\log_a(1+x)}{\log_a e\cdot x}=1 \Rightarrow \log_a(1+x)=x\cdot\log_a e + \overbrace{o(x\cdot \underbrace{\log_a e}_{const})}^{=o(x)}=$

$= x\cdot \log_a e + o(x)$

$\ln(1+x) = x + o(x),\ x\to 0$

3° $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a \Rightarrow a^x-1 = x\ln a + o(x),\ x\to 0$

$a^x = 1 + x\ln a + o(x),\ x\to 0$

$e^x = 1 + x + o(x),\ x\to 0$

4° $\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\alpha \Rightarrow (1+x)^\alpha = 1 + x\cdot\alpha + o(x),\ x\to 0$

$\lim_{x\to a}\frac{f(x)}{g(x)}=c \Rightarrow f(x)=g(x)\cdot c + o(g(x)),\ x\to a$

(1) $a,b,c>0$

$\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x} = \lim_{x\to 0}\left(\frac{1+x\ln a+o(x)+1+x\ln b+o(x)+1+x\ln c+o(x)}{3}\right)^{1/x}=$

$=\lim_{x\to 0}\left(\frac{3+x(\ln a+\ln b+\ln c)+o(x)}{3}\right)^{1/x} = \lim_{x\to 0}\left(1+\underbrace{x\cdot\ln\sqrt[3]{abc}+o(x)}_{\to 0}\right)^{\frac{1}{x\cdot\ln\sqrt[3]{abc}+o(x)}\cdot\frac{A}{x}}$, $A = x\cdot\ln\sqrt[3]{abc}+o(x)$

$\lim_{x\to 0}\frac{A}{x}=\lim_{x\to 0}\frac{x\cdot\ln\sqrt[3]{abc}+o(x)}{x}=\lim_{x\to 0}\ln\sqrt[3]{abc}+\underbrace{o(1)}_{\to 0}=\ln\sqrt[3]{abc}$ $\quad \downarrow e$

$\Rightarrow \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}=e^{\ln\sqrt[3]{abc}}=\sqrt[3]{abc}$

$h=o(c\cdot f),\ x\to a$

$\exists\,\alpha\quad h=\underbrace{\alpha}_{\to 0}\cdot c\cdot f,\quad \alpha\to 0$

$=o(f)$

$o(-f)=o(2f)=o(f)$

(2) $\lim_{x\to 0}\frac{1-(\cos x)^{\sin x}}{x^3}=(*)$

$\cos x = ?$

$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}\Rightarrow 1-\cos x=\frac{1}{2}x^2+o(x^2)$

$\cos x = 1-\frac{1}{2}x^2-o(x^2) = 1-\frac{1}{2}x^2+\underbrace{o(-x^2)}_{h}$

$(*) = \lim_{x\to 0}\frac{1-\left(1-\frac{1}{2}x^2+o(x^2)\right)^{\sin x}}{x^3}=$

$=\lim_{x\to 0}\frac{1-e^{\sin x\cdot\ln(1-\frac{1}{2}x^2+o(x^2))}}{x^3}=(*)$

$$\sin x\cdot \ln\left(1-\frac{1}{2}x^2+o(x^2)\right) = (x+o(x))\left(-\frac{1}{2}x^2+o(x^2)+o\left(-\frac{1}{2}x^2+o(x^2)\right)\right)$$

$\ln(1+x)=x+o(x),\ x\to 0$

$$= -\frac{1}{2}x^3+\underbrace{x\cdot o(x^2)}_{=o(x^3)} + \underbrace{x\cdot \underbrace{o\left(-\frac{1}{2}x^2+o(x^2)\right)}_{o(x^2)}}_{o(x^3)} + \underbrace{o(x)\cdot o(x^2)}_{o(x^3)} + \underbrace{o(x)\cdot o\left(-\frac{1}{2}x^2+o(x^2)\right)}_{o(x^3)}$$

$$\underbrace{o(f+o(f))}_{h} = o(f) \rightsquigarrow o(f+o(f)) = o(f)+o(o(f))$$
$$= o(f)+o(f)=o(f)$$

$$h=\alpha\cdot(f+\underbrace{o(f)}_{=\alpha_2 f})$$
$$=\alpha(f+\alpha_2\cdot f) = \underbrace{(\alpha+\alpha\alpha_2)}_{\to 0} f = o(f)$$

$$= -\frac{1}{2}x^3+o(x^3).$$

$e^t=1+t+o(t),\ t\to 0$

$$(*) = \lim_{x\to 0}\frac{1-e^{\sin x\ln\left(1-\frac{1}{2}x^2+o(x^2)\right)}}{x^3} = \lim_{x\to 0}\frac{1-e^{-\frac{1}{2}x^3+o(x^3)}}{x^3} \overset{\text{''}}{=} \lim_{x\to 0}\frac{1-\left(1+\left(-\frac{1}{2}x^3+o(x^3)\right)+o\left(-\frac{1}{2}x^3+o(x^3)\right)\right)}{x^3}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}x^3+o(x^3)+o\left(-\frac{1}{2}x^3+o(x^3)\right)}{x^3} = \lim_{x\to 0}\frac{\frac{1}{2}x^3+o(x^3)+\underbrace{o\left(-\frac{1}{2}x^3\right)}_{\text{''}o(x^3)}+\underbrace{o(o(x^3))}_{\text{''}o(x^3)}}{x^3}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}x^3+o(x^3)}{x^3} = \lim_{x\to 0}\frac{1}{2}+o(1) = \frac{1}{2}.$$

За вежбу:

1. Наћи асимптотски еквивалентни полином наредним фјама

а) $f(x)=\sqrt{x+1}-\sqrt{1-x},\quad x\to 0$

б) $f(x)=\operatorname{tg}x-\sin x,\quad x\to 0$

в) $f(x)=x^x-1,\quad x\to 1$.

2. $(x+o(x))^n = x^n+o(x^n)$, $x\to 0$ $\rightsquigarrow$ доказати за $n\in\mathbb{N}$.