~ Поднизови и тачке нагомилавања ~

$x : \mathbb{N} \to X$ $\to$ низ

$\varphi : \mathbb{N} \to \mathbb{N}$, $\varphi \uparrow$

$x \circ \varphi$ подниз

$x_n \rightsquigarrow x_{2n}$ $\quad \varphi(n) = 2n$

ово важи у случају да $x \in \mathbb{R}$
ако је $x = \pm\infty$ онда $\forall M>0$ ∃ беск... у $(M, +\infty)$, $(-\infty, -M)$

(x_n) низ. x је тачка нагомилавања ако $\forall \varepsilon > 0$ ∃ бесконачно много чланова низа x_n који се налазе у $(x-\varepsilon, x+\varepsilon)$, односно ∃ подниз $x_{\varphi(n)}$ за који важи $\lim\limits_{n\to\infty} x_{\varphi(n)} = x$.

Т. Болцано–Вајерштрас: Сваки ограничен низ има тачку нагомилавања у $\mathbb{R}$.

Уколико $\exists\, \varphi, \psi : \mathbb{N} \to \mathbb{N}$ $\uparrow$ и $\lim\limits_{n\to\infty} x_{\varphi(n)} \neq \lim\limits_{n\to\infty} x_{\psi(n)}$ онда низ дивергира.

$T(x_n) \to$ скуп тачака нагомилавања низа x_n, $T(x_n) \subseteq \overline{\mathbb{R}}$.

x_n конвергира $\Leftrightarrow$ $T(x_n)$ једночлан и $T(x_n) = \{x\}$ и $x \in \mathbb{R}$.

x_n низ

$$\{x_n \mid n \in \mathbb{N}\} = \bigcup_{i=1}^{k} \{x_{\varphi_i(n)} \mid n \in \mathbb{N}\}$$

$$T(x_n) = \bigcup_{i=1}^{k} T(x_{\varphi_i(n)})$$

1)
$$x_n = (-1)^{n-1}\left(2 + \frac{3}{n}\right)$$

$T(x_n) = ?$

$$\{x_n \mid n \in \mathbb{N}\} = \{x_{2n} \mid n \in \mathbb{N}\} \cup \{x_{2n+1} \mid n \in \mathbb{N}\}$$

$T(x_{2n}) = ?$

$$\left.\begin{array}{ll} x_{2n} = -\left(2 + \frac{3}{n}\right), & \lim\limits_{n\to\infty} x_{2n} = -2 \\ x_{2n+1} = 2 + \frac{3}{n}, & \lim\limits_{n\to\infty} x_{2n+1} = 2 \end{array}\right\} \Rightarrow T(x_n) = \{-2, 2\}$$

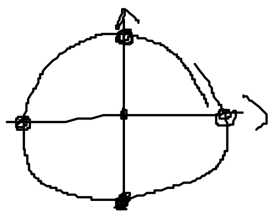
$$\overline{\lim_{n\to\infty}}\, x_n = \sup T(x_n)$$
$$\parallel$$
$$\limsup_{n\to\infty} x_n$$

$$\underline{\lim_{n\to\infty}}\, x_n = \inf T(x_n)$$
$$\parallel$$
$$\liminf_{n\to\infty} x_n$$

$$\overline{\lim_{n\to\infty}}\, x_n = 2$$

$$\underline{\lim_{n\to\infty}}\, x_n = -2$$

(2) $x_n = 1 + \frac{n}{n+1}\cos\frac{n\pi}{2}$



$n = 2k$

$\cos\frac{n\pi}{2} = \cos k\pi$

$n = 4l, \quad l \in \mathbb{N}$

$\cos\frac{n\pi}{2} = \cos 2l\pi = 1 \quad \rightsquigarrow \lim_{n\to\infty} x_{4n} = \lim_{n\to\infty} 1 + \frac{4n}{4n+1}\cdot 1 = 2$

$n = 4l+2$

$\cos\frac{n\pi}{2} = \cos(2l\pi + \pi) = -1 \qquad \lim_{n\to\infty} x_{4n+2} = \lim_{n\to\infty} 1 - \frac{4n+2}{4n+3} = 0$

$n = 2k+1$

$\cos\frac{n\pi}{2} = \cos(k\pi + \frac{\pi}{2}) = 0 \qquad \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} 1 + 0\,\frac{2n+1}{2n+2} = 1$

$T(x_n) = \{0, 1, 2\} \longrightarrow \overline{\lim_{n\to\infty}}\, x_n = 2, \quad \underline{\lim_{n\to\infty}}\, x_n = 0$

(3) Да ли постоји низ тд:

а) $T(x_n) = \overline{\mathbb{R}}$

$\mathbb{Q} = \{r_1, r_2, \dots\} \rightarrow$ низ рационалних бројева

у свакој околини било ког реалног броја $a \in \mathbb{R}$ (или $\pm\infty$) имамо бескон. много рац. бројева јер је $\mathbb{Q}$ густ у $\mathbb{R}$.

б) $T(x_n) = \{b_1, b_2, b_3, \dots, b_k\}$

$x_n = b_l, \quad l \in \mathbb{N} \quad n = m\cdot k + l, \quad l \in \{1, \dots, k\}, \quad m \in \mathbb{N}$

$x_1 = b_1, \ x_{k+1} = b_1, \ x_{2k+1} = b_1 \ \dots$

$x_2 = b_2, \ x_{k+2} = b_2 \ \dots\dots \ x_{mk+2} = b_2 \ \dots$

$\dots$

$\{x_n\} = \bigcup_{l=1}^{k} \{x_{mk+l} \mid m \in \mathbb{N}\} \qquad \varphi_l(m) = mk + l.$

б) $T(x_n) = \mathbb{Q}$?

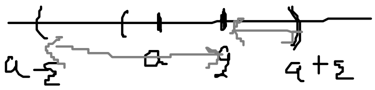
$\mathbb{Q}$ густ у $\mathbb{R}$

$T(x_n) = \mathbb{Q} \Rightarrow \forall q \in \mathbb{Q} \ \forall \varepsilon > 0$ $\exists$ бесков. ш. низа x_n у $(q-\varepsilon, q+\varepsilon)$

$a \in \overline{\mathbb{R}}$, пп. $a \in \mathbb{R}$ $\quad \forall \varepsilon > 0 \quad (a-\varepsilon, a+\varepsilon) \cap \mathbb{Q} \neq \emptyset$

$a = +\infty \qquad (M, +\infty)$

$\exists q \in \mathbb{Q} \quad q \in (a-\varepsilon, a+\varepsilon)$



$\varepsilon_2 = \min\{q-(a-\varepsilon), a+\varepsilon-q\}$ $\quad \varepsilon_2 = q - M > 0$

$(q-\varepsilon_2, q+\varepsilon_2) \subseteq (a-\varepsilon, a+\varepsilon)$

$\Rightarrow$ $\exists$ бесков. ш. низа x_n у $(q-\varepsilon_2, q+\varepsilon_2)$

—||— у $(a-\varepsilon, a+\varepsilon)$

$\Downarrow$

$a \in T(x_n)$

$\Rightarrow T(x_n) = \overline{\mathbb{R}}$

$\nexists x_n : T(x_n) = \mathbb{Q}$

$$\mathbb{Q} \subseteq T(x_n) \Rightarrow \overline{\mathbb{R}} = T(x_n)$$

$$\varlimsup_{n\to\infty} x_n = \sup T(x_n)$$

$$\varliminf_{n\to\infty} x_n = \inf T(x_n)$$

$$\varliminf_{n\to\infty} x_n + \varliminf_{n\to\infty} y_n \leq \varliminf_{n\to\infty}(x_n+y_n) \leq \varliminf_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$$

$$\leq \varlimsup_{n\to\infty}(x_n+y_n) \leq \varlimsup_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$$

$x_n = (-1)^n$

$y_n = (-1)^{n+1}$

$x_n, y_n \geq 0$

$$\varliminf_{n\to\infty} x_n \cdot \varliminf_{n\to\infty} y_n \leq \varliminf_{n\to\infty}(x_n \cdot y_n) \leq \varliminf_{n\to\infty} x_n \cdot \varlimsup_{n\to\infty} y_n \leq \varlimsup_{n\to\infty} x_n \cdot y_n$$

$$\leq \varlimsup_{n\to\infty} x_n \cdot \varlimsup_{n\to\infty} y_n$$

$x_n = 1$ n парно $\qquad \underline{\lim}_{n\to\infty} x_n = 1 = \underline{\lim}_{n\to\infty} y_n$

$x_n = 2$ n непарно

$y_n = 3$ n парно $\qquad \underline{\lim}_{n\to\infty} (x_n \cdot y_n) = 2$

$y_n = 1$ n непарно

$x_n \cdot y_n = 3$ n парно

$x_n \cdot y_n = 2$ n непарно

(4) $x_n > 0$ $\quad \overline{\lim}_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} \frac{1}{x_n} = 1$.

Да ли x_n конвергира?

$a \in T(x_n) \Rightarrow ? \quad \frac{1}{a} \in T\left(\frac{1}{x_n}\right)$?

$\exists\, x_{n_k} : \quad \lim_{k\to\infty} x_{n_k} = a$

$$\Rightarrow \lim_{k\to\infty} \frac{1}{x_{n_k}} = \frac{1}{a} \Rightarrow \frac{1}{a} \in T\left(\frac{1}{x_n}\right)$$

$b \in T\left(\frac{1}{x_n}\right) \Rightarrow \frac{1}{b} \in T\left(\frac{1}{\frac{1}{x_n}}\right) = T(x_n)$

$$\Rightarrow T(x_n) = \frac{1}{T\left(\frac{1}{x_n}\right)}$$

$$\overline{\lim}_{n\to\infty} \frac{1}{x_n} = \frac{1}{\underline{\lim}_{n\to\infty} x_n}, \text{ јер } \underline{\underline{x_n > 0}}$$

$$\Rightarrow 1 = \overline{\lim}_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} \frac{1}{x_n} = \frac{\overline{\lim}_{n\to\infty} x_n}{\underline{\lim}_{n\to\infty} x_n} \Rightarrow \overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$$

$\Rightarrow T(x_n) \subseteq \mathbb{R}$ и

$T(x_n) = \{x\}$

$T(x_n) = \{x\} \Rightarrow x_n$ конвергира
$x \in \mathbb{R}$

⑥ Испитати конв низа $\sin n, n \in \mathbb{N}$.

$x_n = \sin n$

ПП x_n конвергира, $\lim\limits_{n\to\infty} x_n = x \in [-1,1]$

$\sin(n+1) = \sin n \cdot \cos 1 + \sin 1 \cos n$

$\sin 2n = 2\sin n \cdot \cos n \quad /^2$

$\sin^2 2n = 4\sin^2 n \cdot \cos^2 n = 4\sin^2 n(1-\sin^2 n) = 4\sin^2 n - 4\sin^4 n \quad / \lim\limits_{n\to\infty}$

$(\lim\limits_{n\to\infty} \sin 2n)^2 = x^2$

$x^2 = 4x^2 - 4x^4 \rightsquigarrow 0 = 3x^2 - 4x^4 = x^2(3-4x^2)$

$x \in \{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\} = A$

$\sin 3n = \underbrace{\sin 2n}_{2\sin n\cos n} \cos n + \sin n \cdot \underbrace{\cos 2n}_{\cos^2 n - \sin^2 n = 1 - 2\sin^2 n}$

$= 2\sin n \cdot (1-\sin^2 n) + \sin n(1-2\sin^2 n) =$

$= 3\sin n - 4\sin^3 n \quad / \lim\limits_{n\to\infty}$

$x = 3x - 4x^3 \rightsquigarrow 0 = 2x(1-2x^2)$

$x \in \{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\} = B$

$x \in A \cap B = \{0\}$

$x = 0$

$\sin(n+1) = \sin n \cdot \cos 1 + \sin 1 \cos n \quad / \lim\limits_{n\to\infty}$

$0 = \lim\limits_{n\to\infty} \sin(n+1) = \underbrace{\lim\limits_{n\to\infty} \sin n \cdot \underbrace{\cos 1}_{\neq 0}}_{=0} + \sin 1 \lim\limits_{n\to\infty} \cos n$

$\Rightarrow \lim\limits_{n\to\infty} \cos n = 0$

$\sin^2 n + \cos^2 n = 1 \quad / \lim\limits_{n\to\infty} \quad \Rightarrow \quad 0 = 1 \quad ↯$

$\Rightarrow$ $\sin n$ дивергира.

Испитати конв низа $\sin nx$, $x \in \mathbb{R}$.