Штолцова теорема

a_n, b_n , $b_n \uparrow$ и $\lim\limits_{n\to\infty} b_n = +\infty \Rightarrow \lim\limits_{n\to\infty} \frac{a_n}{b_n} \overset{⊛}{=} \lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$, ⊛ - уколико десна страна постоји

① $\lim\limits_{n\to\infty} \frac{1^p+2^p+3^p+\dots+n^p}{(n+1)^p} = ?$ $\quad p>0$

$b_n = (n+1)^p \uparrow \quad n\in\mathbb{N}, p>0 \quad \lim\limits_{n\to\infty} b_n = +\infty$

$a_n = 1^p+2^p+\dots+n^p$

$$\lim_{n\to\infty}\frac{a_n}{b_n} \overset{⊛}{=} \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{(n+1)^p}{(n+2)^p-(n+1)^p} = \lim_{n\to\infty}\frac{1}{\left(\frac{n+2}{n+1}\right)^p-1} = \lim_{n\to\infty}\frac{1}{\underbrace{\underbrace{\left(1+\frac{1}{n+1}\right)^p}_{\downarrow \atop 1+0}-1}_{\to 0^+}} = +\infty$$

$\lim\limits_{n\to\infty}\frac{a_n}{b_n} = +\infty$ ✓

② $\lim\limits_{n\to\infty} \frac{1^k+2^k+\dots+n^k}{n^{k+1}} = ?$ $\quad k\in\mathbb{N}$

$a_n = 1^k+2^k+\dots+n^k$

$b_n = n^{k+1}$

$b_n \uparrow$, $\lim\limits_{n\to\infty} n^{k+1} = +\infty$

$$\lim_{n\to\infty}\frac{1^k+\dots+n^k}{n^{k+1}} \overset{⊛}{\underset{ШТ}{=}} \lim_{n\to\infty}\frac{(1^k+\dots+n^k+(n+1)^k)-(1^k+\dots+n^k)}{(n+1)^{k+1}-n^{k+1}} = \lim_{n\to\infty}\frac{(n+1)^k}{(n+1)^{k+1}-n^{k+1}}$$

$$= \lim_{n\to\infty}\frac{\sum\limits_{m=0}^{k}\binom{k}{m}n^m}{\sum\limits_{m=0}^{k+1}\binom{k+1}{m}n^m - n^{k+1}} = \lim_{n\to\infty}\frac{\sum\limits_{m=0}^{k}\binom{k}{m}n^m}{\sum\limits_{m=0}^{k}\binom{k+1}{m}n^m} = \frac{\binom{k}{k}}{\binom{k+1}{k}} = \frac{1}{k+1}$$

$\hookrightarrow m=k+1 \quad \binom{k+1}{k+1}=1$

$$\lim_{n\to\infty}\frac{1^k+\dots+n^k}{n^{k+1}} = \frac{1}{k+1}$$

③ Кошијева теорема : $\lim\limits_{n\to\infty} a_n = a\in\mathbb{R} \Rightarrow \lim\limits_{n\to\infty}\frac{a_1+a_2+\dots+a_n}{n} = a$

$b_n = n \uparrow$, $\lim\limits_{n\to\infty} b_n = +\infty$

$$\lim_{n\to\infty}\frac{a_1+\dots+a_n}{n} \overset{ШТ}{\underset{⊛}{=}} \lim_{n\to\infty}\frac{a_{n+1}}{1} = a .$$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n]{a_1\cdot a_2\cdots a_n} = a = \lim_{n\to\infty}\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}} = \lim_{n\to\infty}\sqrt[k]{\frac{a_1^k+\dots a_n^k}{n}}$$

T. $\forall n \in \mathbb{N}\ a_n > 0$
$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a_n} \overset{(*)}{=} \lim_{n\to\infty} \frac{a_n}{a_{n-1}}$$, $(*)$ – уколико десна страна постоји

(4) $$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} \underset{T.}{\overset{(*)}{=}} \lim_{n\to\infty} \frac{\frac{n!}{n^n}}{\frac{(n-1)!}{(n-1)^{n-1}}} = \lim_{n\to\infty} \frac{(n-1)^{n-1}}{n^{n-1}} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n-1} =$$

$$= \lim_{n\to\infty} \Bigg(\underbrace{\left(1-\frac{1}{n}\right)^{-n}}_{\to e}\Bigg)^{\overbrace{-\frac{n-1}{n}}^{\to -1}} = \frac{1}{e}$$

$$\lim_{n\to\infty} a_n = \pm\infty \Rightarrow \lim_{n\to\infty} \left(1+\frac{1}{a_n}\right)^{a_n} = e.$$

за вежбу одредити $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$

(5) $$\lim_{n\to\infty} \frac{n}{a^n} \sum_{k=1}^{n} \frac{a^{k-1}}{k} = ? \quad , \quad a>1$$

$$L = \lim_{n\to\infty} \frac{n \cdot \sum_{k=1}^{n} \frac{a^{k-1}}{k}}{a^n}$$

$$(n+1)\sum_{k=1}^{n+1} \frac{a^{k-1}}{k} - n\sum_{k=1}^{n} \frac{a^{k-1}}{k} = \sum_{k=1}^{n+1} \frac{a^{k-1}}{k} + n\cdot\frac{a^n}{n+1}$$ $\rightsquigarrow$ није баш најбоље

$$L = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{a^{k-1}}{k}}{\frac{a^n}{n}} \underset{ШТ}{\overset{(*)}{=}} \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} \frac{a^{k-1}}{k} - \sum_{k=1}^{n} \frac{a^{k-1}}{k}}{\frac{a^{n+1}}{n+1} - \frac{a^n}{n}} = \lim_{n\to\infty} \frac{\frac{a^n}{n+1}}{\frac{a^{n+1}}{n+1} - \frac{a^n}{n}} = \lim_{n\to\infty} \frac{1}{a - \frac{n+1}{n}} = \frac{1}{a-1} \checkmark$$

$\frac{a^n}{n} \to +\infty$

$\frac{a^n}{n} \uparrow$ за $n \geq n_0$ $\qquad \Rightarrow L = \frac{1}{a-1}$.

(6) $x_0 > 0$, $x_{n+1} = \frac{x_n}{1+x_n^2}$ $\qquad \lim_{n\to\infty} x_n = ?$

$\qquad \lim_{n\to\infty} n \cdot x_n^2 = ?$

$x_n \ \square\ x_{n+1}$

$x_n \ \square\ \frac{x_n}{1+x_n^2} \quad / \cdot \underbrace{(1+x_n^2)}_{>0}$

$\cancel{x_n} + x_n^3 \ \square\ \cancel{x_n}$

$x_n^3 \ \square\ 0$

$x_0 > 0 \Rightarrow \forall n \in \mathbb{N}\ x_n > 0 \qquad \Rightarrow x_n > x_{n+1}$

ПМИ: ... $\qquad$ $x_n \downarrow$

$x_n \downarrow$ и $x_n > 0 \Rightarrow x_n$ конвертира

$$\lim_{n\to\infty} x_n = x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n}{1+x_n^2} = \frac{x}{1+x^2}$$

$$x = \frac{x}{1+x^2} \Rightarrow x\cdot\left(1 - \frac{1}{1+x^2}\right) = 0 \Rightarrow \underline{\underline{x=0}}$$

$$\lim_{n\to\infty} x_n = 0.$$

$$\lim_{n\to\infty} n\cdot x_n^2 = ?$$

$$x_{n+1} = \frac{x_n}{1+x_n^2}, \quad x_0 > 0$$

$$\lim_{n\to\infty} n\cdot x_n^2 = \lim_{n\to\infty} \frac{n}{\frac{1}{x_n^2}} \overset{ШТ}{\underset{*}{=}} \lim_{n\to\infty} \frac{n+1-n}{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\frac{x_n^2}{(1+x_n^2)^2}} - \frac{1}{x_n^2}} = \lim_{n\to\infty} \frac{1}{\frac{(1+x_n^2)^2-1}{x_n^2}} =$$

$x_n \downarrow \Rightarrow \frac{1}{x_n^2} \uparrow$ и $\lim\limits_{\substack{n\to\infty\\ x_n\to 0}} \frac{1}{x_n^2} = +\infty$

$$= \lim_{n\to\infty} \frac{x_n^2}{x_n^4 + 2x_n^2} = \lim_{n\to\infty} \frac{1}{x_n^2+2} = \frac{1}{2}$$

7) $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$, $x, y \in \mathbb{R}$

$$\lim_{n\to\infty} \underbrace{\frac{x_1\cdot y_n + x_2\cdot y_{n-1} + \dots + x_n\cdot y_1}{n}}_{z_n} = ?$$

1° $x = y = 0$

$$0 \le \left|\frac{x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1}{n}\right| \le \frac{|x_1|\overset{\le M}{|y_n|} + \dots + |x_n|\overset{\le M}{|y_1|}}{n} \le M\frac{(|x_1| + \dots + |x_n|)}{n}$$

$\downarrow$ штолцова теорема $n\to+\infty$

$M\cdot 0 = 0$

$$\lim_{n\to\infty} z_n = 0 \Leftrightarrow \lim_{n\to\infty} |z_n| = 0$$

$$\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n \Rightarrow \lim_{n\to\infty} |y_n| = \lim_{n\to\infty} |x_n| = 0$$

$$\Rightarrow \exists M > 0 \ \forall n \in \mathbb{N} \ |y_n| \le M$$

Тогп $\Rightarrow$ $\lim\limits_{n\to\infty} |z_n| = 0 \Rightarrow \lim\limits_{n\to\infty} \frac{x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1}{n} = 0$

2° $x, y \in \mathbb{R}$

$\lim\limits_{n\to\infty} x_n = x \Rightarrow \lim\limits_{n\to\infty} (\underbrace{x_n - x}_{\alpha_n}) = 0 \Rightarrow \lim\limits_{n\to\infty} \alpha_n = 0$, $x_n = x + \alpha_n$

$\lim\limits_{n\to\infty} y_n = y$ $\quad \lim\limits_{n\to\infty} (\underbrace{y_n - y}_{\beta_n}) = 0 \Rightarrow \lim\limits_{n\to\infty} \beta_n = 0$, $y_n = y + \beta_n$

$$x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1 = (x+\alpha_1)(y+\beta_n) + (x+\alpha_2)(y+\beta_{n-1}) + \dots + (x+\alpha_n)(y+\beta_1)$$

$$= xy + x\cdot\beta_n + y\cdot\alpha_1 + \alpha_1\beta_n + xy + x\cdot\beta_{n-1} + y\alpha_2 + \alpha_2\beta_{n-1} + \dots + xy + y\alpha_n + x\beta_1 + \alpha_n\beta_1 =$$

$$= n\cdot xy + x(\beta_1+\beta_2+\dots+\beta_n) + y(\alpha_1+\dots+\alpha_n) + \alpha_1\beta_n + \alpha_2\beta_{n-1} + \dots + \alpha_n\beta_1$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{nxy + x(\beta_1+\dots+\beta_n) + y(\alpha_1+\dots+\alpha_n) + \alpha_1\beta_n + \alpha_2\beta_{n-1} + \dots + \alpha_n\beta_1}{n}$$

$$= xy + x\underbrace{\lim_{n\to\infty}\frac{\beta_1+\beta_2+\dots+\beta_n}{n}}_{\substack{=0 \\ \text{ШТ}}} + y\underbrace{\lim_{n\to\infty}\frac{\alpha_1+\dots+\alpha_n}{n}}_{\substack{=0 \\ \text{ШТ}}} + \underbrace{\lim_{n\to\infty}\frac{\alpha_1\beta_n+\alpha_2\beta_{n-1}+\dots+\alpha_n\beta_1}{n}}_{\substack{= 0 \\ 1^\circ}} = x\cdot y$$