③ $a_n = \left(1+\frac{1}{n}\right)^n$, $b_n = \left(1+\frac{1}{n}\right)^{n+1}$ $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = e$

$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$$

$\lim\limits_{n\to\infty} x_n = e$.

$x_n \geq a_n \quad \forall n\in\mathbb{N} \implies e \leq \lim\limits_{n\to\infty} x_n$

$\forall k\in\mathbb{N} \quad x_k \leq e = \lim\limits_{n\to\infty} a_n$?

k - фиксирамо

$\underline{n>k}$ $\quad a_n = \left(1+\frac{1}{n}\right)^n = \sum\limits_{l=0}^{n} \binom{n}{l} \frac{1}{n^l} = \sum\limits_{l=0}^{n} \frac{n!}{l!\cdot(n-l)!} \cdot \frac{1}{n^l} =$

$$= \sum_{l=0}^{n} \frac{\overbrace{n\cdot(n-1)(n-2)\dots(n-l+1)}^{l}}{l!} \cdot \frac{1}{n^l} =$$

$$= \sum_{l=0}^{n} \underbrace{\frac{1}{l!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-l+1}{n}}_{>0} \underset{n>k}{>} \sum_{l=0}^{k} \frac{1}{l!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-l+1}{n}$$

$$a_n > \underbrace{\frac{1}{0!} + \frac{1}{1!}\cdot 1 + \frac{1}{2!}\cdot\frac{n}{n}\cdot\frac{n-1}{n} + \dots + \frac{1}{k!}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n}}_{k+1 \text{ - сабирак}} \quad \Big/ \lim_{n\to\infty}$$

$$\implies \lim_{n\to\infty} a_n \geq \lim_{n\to\infty} \Big(\underbrace{\frac{1}{0!} + \frac{1}{1!}\cdot 1 + \dots + \frac{1}{k!}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n}}_{\text{број сабирака не зависи од } n} \Big) =$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!}\cdot \underbrace{\lim_{n\to\infty}\frac{n}{n}\cdot\frac{n-1}{n}}_{=1} + \frac{1}{3!}\underbrace{\lim_{n\to\infty}\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}}_{=1} + \dots + \frac{1}{k!}\underbrace{\lim_{n\to\infty}\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n}}_{=1}$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!} = x_k$$

$\implies x_k \leq e \; , \; \forall k\in\mathbb{N} \implies \lim\limits_{k\to\infty} x_k \leq e$

То2п
$\implies \lim\limits_{k\to\infty} x_k = e$.

(4) $\underset{\shortparallel}{e} - x_n < \frac{1}{n \cdot n!} \qquad n \in \mathbb{N}$

$\lim\limits_{m\to\infty} x_m$, $m > n$, n фиксирано $\Rightarrow m = n+k$, $k \in \mathbb{N}$

$$x_m - x_n = x_{n+k} - x_n = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(n+k)!}\right) - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right)$$

$$= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{1}{(n+k)!} = \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots + \frac{1}{(n+2)\dots(n+k)}\right)$$

$$\le \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \dots + \frac{1}{(n+2)^{k-1}}\right) = *$$

$$\downarrow$$

$$\frac{1}{n+l} \le \frac{1}{n+2} \quad , l \ge 2$$

$$a = \frac{1}{n+2} < 1$$

$$* = \frac{1}{(n+1)!}\left(1 + a + a^2 + a^3 + \dots + a^{k-1}\right) = \frac{1}{(n+1)!} \cdot \frac{1-a^k}{1-a} =$$

$$= \frac{1}{(n+1)!} \cdot \frac{1 - \left(\frac{1}{n+2}\right)^k}{1 - \frac{1}{n+2}} = \frac{1}{(n+1)!} \cdot \frac{n+2}{n+1} \cdot \left(1 - \frac{1}{(n+2)^k}\right)$$

$$\Rightarrow x_{n+k} - x_n \le \frac{1}{(n+1)!} \cdot \frac{n+2}{n+1} \cdot \left(1 - \underbrace{\frac{1}{(n+2)^k}}_{\to 0}\right) \quad \Big| \lim_{k\to\infty}$$

$$\Rightarrow e - x_n \le \underbrace{\frac{1}{(n+1)!}}_{= n! \cdot (n+1)} \cdot \frac{n+2}{n+1} \cdot 1 \overset{?}{<} \frac{1}{n} \cdot \frac{1}{n!}$$

$$? \ \frac{n+2}{(n+1)^2} < \frac{1}{n} \ ? \qquad (n+2) \cdot n < (n+1)^2 = n^2 + 2n + 1 \ \checkmark$$

$$\Rightarrow e - x_n < \frac{1}{n \cdot n!}$$

(5) Доказати да је e ирационалан.

Претпоставимо супротно, e је рационалан

$\Rightarrow \exists m \in \mathbb{Z}, n \in \mathbb{N} \quad e = \frac{m}{n} > x_k \quad \forall k \in \mathbb{N} \Rightarrow \underline{\underline{m \in \mathbb{N}}}$

$$\left.\begin{array}{l} 0 < e - x_n < \frac{1}{n \cdot n!} \\ \underset{\in \mathbb{Q}}{e} - \underset{\in \mathbb{Q}}{x_n} \in \mathbb{Q} \end{array}\right\} \Rightarrow e - x_n = \frac{q}{n \cdot n!} \quad , q \in (0,1) \cap \mathbb{Q}$$

$$\frac{m}{n} = e = x_n + \frac{\vartheta}{n \cdot n!} = 1 + \frac{1}{1!} + \dots + \frac{1}{n!} + \frac{\vartheta}{n \cdot n!} \qquad / \cdot n!$$

$$\underbrace{m \cdot (n-1)!}_{\in \mathbb{Z}} = \underbrace{n! \cdot \left(1 + \frac{1}{1!} + \dots + \frac{1}{n!}\right)}_{\in \mathbb{Z}} + \underbrace{\frac{\vartheta}{n}}_{\in (0,1)} , \qquad \vartheta \in (0,1)$$

$$\Rightarrow \underbrace{m \cdot (n-1)! - n! \cdot x_n}_{\in \mathbb{Z}} \in (0,1) \quad \lightning \quad \Rightarrow e \text{ није рационалан}$$

$$\left(1 + \frac{1}{n}\right)^n \xrightarrow[n \to +\infty]{} e$$

Т. $\lim_{n \to \infty} a_n = \pm\infty \Rightarrow \lim_{n \to +\infty} \left(1 + \frac{1}{a_n}\right)^{a_n} = e$.

нпр.

$$\lim_{n \to +\infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \to \infty} \left(\left(1 - \frac{1}{n}\right)^{-n}\right)^{-1} = \frac{1}{e}$$

6) $\lim_{n \to \infty} \underbrace{\frac{n!}{n^n}}_{a_n} = 0$:

$$\frac{a_n}{a_{n+1}} = \frac{\frac{\cancel{n!}}{n^n}}{\frac{\cancel{(n+1)!}}{(n+1)^{n+1}}} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \longrightarrow e > 1$$

$$\Rightarrow \exists n_0 \in \mathbb{N} \quad \frac{a_n}{a_{n+1}} > 1 \quad \text{за } n \geq n_0$$

$$\Rightarrow a_n \downarrow, \ a_n > 0 \quad \Rightarrow a_n \text{ конвергира} \quad \text{и} \quad \lim_{n \to \infty} a_n = a$$

$$a = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n \cdot \left(1 + \frac{1}{n}\right)^{-n} = a \cdot \underbrace{\overbrace{\frac{1}{e}}^{>0}}_{<1} \Rightarrow a = 0$$

7) $\lim_{n \to \infty} \underbrace{\frac{n^n}{(n!)^2}}_{a_n} = ?$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{n^n}{(n!)^2}}{\frac{(n+1)^{n+1}}{[(n+1)!]^2}} = \frac{n^n}{(n+1)^{n+1}} \cdot (n+1)^2 = \frac{n^n}{(n+1)^{n-1}} = n \cdot \left(\frac{n}{n+1}\right)^{n-1}$$

$$= n \cdot \left(1 - \frac{1}{n+1}\right)^{n-1} \geq 1 \qquad n \geq n_0$$

$$\lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right)^{n-1} = \lim_{n \to \infty} \underbrace{\left(1 + \frac{1}{-(n+1)}\right)^{-(n+1)}}_{\to e} {}^{\overbrace{\frac{n-1}{-(n+1)}}^{\to -1}} = e^{-1} = \frac{1}{e} > 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = +\infty \Rightarrow \exists n_0 \in \mathbb{N} \quad a_n \downarrow \quad n \geq n_0$$

$$\frac{a_n}{a_{n+1}} > 1$$

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = a \cdot \lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\to 0} \cdot \underbrace{\left(1 - \frac{1}{n+1}\right)^{-(n-1)}}_{\to e} = 0 .$$

$$\Rightarrow \lim_{n\to\infty} \frac{n^n}{(n!)^2} = 0$$

$$\boxed{\log_a n \ll n^k \ll c^n \ll n! \ll n^n \ll (n!)^2}$$

* Кошијев критеријум конвергенције

(a_n) Кошијев ако $(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n, m \in \mathbb{N})$ $n, m \geq n_0 \Rightarrow |a_n - a_m| < \varepsilon$.

T. (a_n) Кошијев акко (a_n) конвергира.

(1) Испитати конвергентност низова:

a) $a_n = \frac{\cos(1!)}{1\cdot 2} + \frac{\cos(2!)}{2\cdot 3} + \dots + \frac{\cos(n!)}{n\cdot(n+1)}$

Да ли је (a_n) Кошијев?

$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad m, n \geq n_0 \quad |a_m - a_n| < \varepsilon$?

$\varepsilon > 0$ произвољно, $n_0 = ?$ $\quad m, n \geq n_0 \quad |a_m - a_n| < \varepsilon$.

$|a_m - a_n| < \varepsilon$

$m > n$

$$|a_m - a_n| = \left| \frac{\cos(m!)}{m\cdot(m+1)} + \frac{\cos(m-1)!}{m\cdot(m-1)} + \dots + \frac{\cos(n+1)!}{(n+1)(n+2)} \right| \underset{\text{хоћемо}}{<} \varepsilon$$

$$|a_m - a_n| \leq \frac{\overbrace{|\cos m!|}^{\leq 1}}{m\cdot(m+1)} + \frac{\overbrace{|\cos(m-1)!|}^{\leq 1}}{(m-1)\cdot m} + \dots + \frac{\overbrace{|\cos(n+1)!|}^{\leq 1}}{(n+1)(n+2)}$$

$$\leq \frac{1}{m\cdot(m+1)} + \frac{1}{(m-1)\cdot m} + \dots + \frac{1}{(n+1)(n+2)} =$$

$$= \cancel{\frac{1}{m}} - \frac{1}{m+1} + \cancel{\frac{1}{m-1}} - \cancel{\frac{1}{m}} + \frac{1}{m-2} - \cancel{\frac{1}{m-1}} \dots + \cancel{\frac{1}{n+2}} - \frac{1}{n+3} + \frac{1}{n+1} - \cancel{\frac{1}{n+2}} =$$

$$= \frac{1}{n+1} - \frac{1}{m+1} \underset{m>n}{<} \frac{1}{n+1} \leq \varepsilon$$

$\Rightarrow n+1 \geq \frac{1}{\varepsilon}$, довољно $n \geq \frac{1}{\varepsilon}$ $\Rightarrow n_0 = \left[\frac{1}{\varepsilon}\right] + 1$ узмемо да важи ово

$\Rightarrow n, m \geq n_0 \Rightarrow |a_n - a_m| < \varepsilon$.

$\Rightarrow (a_n)$ Кошијев $\Leftrightarrow (a_n)$ конвергентан.

б) $b_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$, $b_n \uparrow$, $b_n > 1$

$m > n$

$b_m - b_n = \frac{1}{m} + \frac{1}{m-1} + \dots + \frac{1}{n+1} \xrightarrow[m,n \to +\infty]{} 0$?

$m = 2n \Rightarrow b_{2n} - b_n = \frac{1}{2n} + \frac{1}{2n-1} + \dots + \frac{1}{n+1} \geq \frac{1}{2}$

ПМИ: $n = 1$

$b_2 - b_1 = \frac{1}{2}$ ✓

ик) $n \Rightarrow n+1$?

$\underbrace{\frac{1}{2n} + \frac{1}{2n-1} + \dots + \frac{1}{n+1}}_{B} \geq \frac{1}{2}$? $\underbrace{\frac{1}{2n+2} + \frac{1}{2n+1} + \frac{1}{2n} + \dots + \frac{1}{n+2}}_{A} \geq \frac{1}{2}$?

$A = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} + B \geq \frac{1}{2} + \underbrace{\frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}}_{\geq 0 \ ✓} \geq \frac{1}{2}$?

$\varepsilon = \frac{1}{2}$ $n_0 \in \mathbb{N}$ произвољно $m = 2n_0$, $n = n_0 \Rightarrow |b_m - b_n| \geq \frac{1}{2}$

$\Rightarrow (b_n)$ није Кошијев $\Rightarrow (b_n)$ не конвергира. $b_n \uparrow$ $\Rightarrow \lim\limits_{n \to \infty} b_n = +\infty$

в) $c_n = b_n - \ln n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$

c_n монотон?

$c_{n+1} - c_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+1} - \ln(n+1) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n\right)$

$= \frac{1}{n+1} - \ln(n+1) + \ln n = \frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right) = \underbrace{\frac{1}{n+1} - \ln\left(1+\frac{1}{n}\right)}_{<0}$

$\left(1+\frac{1}{n}\right)^n < e \quad / \ln \quad \Rightarrow n \cdot \ln\left(1+\frac{1}{n}\right) < 1$

$\ln(1 + 1/n) < 1/n$

$\left(1+\frac{1}{n}\right)^{n+1} > e \quad / \ln \quad \Rightarrow (n+1) \cdot \ln(1 + 1/n) > 1$

$\ln(1 + 1/n) > \frac{1}{n+1}$

$c_n \downarrow$

Ако c_n ограничен онда он конвергира.

$$1+\frac{1}{k}=\frac{k+1}{k}$$

$c_n > ?$

$$c_n = 1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n > \ln\left(1+\frac{1}{1}\right)+\ln\left(1+\frac{1}{2}\right)+\dots\ln\left(1+\frac{1}{n}\right)-\ln n$$

$$= \ln\left(\frac{\cancel{2}}{1}\cdot\frac{\cancel{3}}{\cancel{2}}\cdots\frac{n+1}{\cancel{n}}\right)-\ln n = \ln(n+1)-\ln n > 0 \quad \checkmark \ \forall n\in\mathbb{N}$$

$\Rightarrow$ c_n ограничен

$\Rightarrow$ c_n конвергира $\Rightarrow$ c_n Кошијев

$\lim\limits_{n\to\infty} c_n = \gamma$ – Ојлерова константа

$\gamma = 0{,}57\ldots > \frac{1}{2}$