Т. a_n монотон, ограничен $\Rightarrow$ a_n конвергира

a_n монотон, неограничен $\Rightarrow$ a_n дивергира

$a_n \uparrow$, неогран $\Rightarrow$ $\lim\limits_{n\to\infty} a_n = +\infty$

$a_n \downarrow$, неогран $\Rightarrow$ $\lim\limits_{n\to\infty} a_n = -\infty$

(1.) $a_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}_{n} > 0$

$a_n \uparrow$: $a_{n+1} > a_n$?

$a_{n+1} - a_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{\cdots\sqrt{2}}}}}_{n+1} - a_n$

$a_{n+1} = \sqrt{2+a_n} > a_n$? $\uparrow^2$

$2 + a_n > a_n^2$?

$a_n^2 - a_n - 2 < 0$? $a_n \in (x_1, x_2)$.

$x^2 - x - 2 = 0 \Rightarrow x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} \rightarrow \begin{matrix} x_1 = -1 \\ x_2 = 2 \end{matrix}$

$a_{n+1} > a_n$ акко $a_n \in (-1, 2)$

Kada je $a_n < 2$?

(БИ) $n=1$ $a_1 = \sqrt{2} < 2$

(ИК) $a_{n-1} < 2 \Rightarrow a_n = \sqrt{2+a_{n-1}} < 2$ $\forall n \geq 2$.

ПМИ $\Rightarrow$ $a_n < 2$ и $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$.

$a_n \uparrow$ и $\forall n \in \mathbb{N}$ $a_n < 2$ $\overset{T}{\Rightarrow}$ a_n конвергира

$\overset{\uparrow^2}{\underset{\geq 0}{a}} = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+a} \Rightarrow \underline{\underline{a = 2}}$

(2) $x_1 = a > 0$

$x_{n+1} = x_n + \frac{1}{x_n}$?

· $x_n > 0$ $\forall n \in \mathbb{N}$

(БИ) $n=1$ $x_1 = a > 0$ ✓

(ИК) $n \in \mathbb{N}$: $x_n > 0$, $x_{n+1} > 0$?

$x_{n+1} = x_n + \frac{1}{x_n} > x_n > 0$

$\Rightarrow x_{n+1} = x_n + \underset{\substack{\vee \\ 0}}{\frac{1}{x_n}} > x_n \Rightarrow x_n \uparrow$ $\qquad \forall n \in \mathbb{N} \;\; x_n \geq a$

$x = \lim\limits_{n\to\infty} x_n \geq a > 0$

Ако x_n конв ка x онда је $x = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{n+1} =$

$= \lim\limits_{n\to\infty}\left(x_n + \frac{1}{x_n}\right) = x + \frac{1}{x}$

тј. $\frac{1}{x} = 0 \Rightarrow x$ не постоји.

$\Rightarrow x_n$ дивергира, $x_n \uparrow \Rightarrow \lim\limits_{n\to\infty} x_n = +\infty$

③ $c > 0$, $a_n = \frac{c^n}{n!}$, $\lim\limits_{n\to\infty} a_n = ?$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{c^n}{n!}}{\frac{c^{n+1}}{(n+1)!}} = \frac{n+1}{c} \geq 1 \quad \text{за} \quad c \leq n+1$$

$\underline{\underline{n \geq c-1}} \Rightarrow a_n \geq a_{n+1}$

за $n \geq [c-1] \Rightarrow a_n \downarrow$

a_n ограничен? $\quad a_n > 0$

$\overset{T}{\Rightarrow} a_n$ конвергира $\lim\limits_{n\to\infty} a_n = a \geq 0$

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n \cdot \frac{c}{n+1} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{c}{n+1} = a \cdot 0 = 0.$$

$$\boxed{\lim_{n\to\infty} \frac{c^n}{n!} = 0} \qquad c^n \ll n!$$

④ $\lim\limits_{n\to\infty} \frac{n^k}{c^n} = 0$, $k > 0$, $c > 1$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{n^k}{c^n}}{\frac{(n+1)^k}{c^{n+1}}} = c \cdot \Big(\underbrace{\frac{n}{n+1}}_{<1}\Big)^k$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = c \underbrace{\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k}_{\to 1} = c > 1 \Rightarrow \varepsilon = c-1 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0$$

$$\left|\frac{a_n}{a_{n+1}} - c\right| < c-1$$

$$\frac{a_n}{a_{n+1}} \in (c-(c-1),\; c+c-1)$$

$$= (1, 2c-1) \subseteq (1, +\infty)$$

$\Rightarrow \frac{a_n}{a_{n+1}} > 1$ за $n \geq n_0$ $\Rightarrow$ $n \geq n_0$ $a_n \downarrow$

$a_n = \frac{n^k}{c^n} > 0$

$\Rightarrow a_n$ конвергира

$a = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \underbrace{\left(\frac{1}{c} \cdot \left(\frac{n}{n+1}\right)^{-k} \cdot a_n \right)}_{a_{n+1}} = a \cdot \underbrace{\frac{1}{c}}_{<1}$, $\left(\frac{n}{n+1}\right) \to 1$

$\Rightarrow a \cdot \underbrace{\left(1 - \frac{1}{c}\right)}_{\neq 0} = 0 \Rightarrow a = 0.$

$\boxed{\lim\limits_{n\to\infty} \frac{n^k}{c^n} = 0}$ $\quad n^k \ll c^n \ll n! \Rightarrow \lim\limits_{n\to\infty} \frac{n^k}{n!} = 0$

(5) $\lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$, $a > 1$

$\varepsilon > 0$ произвољно $n_0 = ?$ $n \geq n_0$ $\left| \underbrace{\frac{\log_a n}{n}}_{\geq 0} - 0 \right| < \varepsilon$ $\quad \| \longrightarrow$ јер $a > 1$ и $n \geq 1$

$\frac{\log_a n}{n} < \varepsilon$

$\frac{\log_a n}{n} < \varepsilon \iff \log_a n < n \cdot \varepsilon$ $\uparrow a > 1$

$\iff a^{\log_a n} < a^{n\varepsilon}$

$n < (a^\varepsilon)^n$, $n \geq ?$

за $k = 1$, $c = a^\varepsilon > 1$, $\varepsilon > 0$, $a > 1$

$\frac{n}{(a^\varepsilon)^n} < 1$?

$\frac{n}{(a^\varepsilon)^n} \to 0$, $n \to +\infty$

(4) $\Rightarrow$ за $\varepsilon_1 = 1$ $\exists \underline{\underline{n_0}} \in \mathbb{N}$ $\forall n \geq n_0$ $\frac{n}{(a^\varepsilon)^n} < 1$.

$\Rightarrow \boxed{\lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0}$ $\quad \boxed{\log_a n \ll n^k}$, $k > 0$

$\boxed{\log_a n \ll n^k \ll c^n \ll n!}$

⑥ $\lim_{n\to\infty} \sqrt[n]{a^n + n^k} = ?$ $a > 1$, $k > 0$

$\sqrt[n]{a^n} \leq \sqrt[n]{a^n + n^k} \leq \sqrt[n]{a^n + a^n} = a \cdot \sqrt[n]{2}$

$\sqrt[n]{a^n} = a$

$\sqrt[n]{a^n+n^k} \xrightarrow{\text{Тоzn}} a$, $a\cdot\sqrt[n]{2} \xrightarrow{n\to\infty} a$

$\frac{n^k}{a^n} \to 0$, $n\to\infty$

$n \geq n_0 \Rightarrow n^k \leq a^n$

$$\lim_{n\to\infty} \sqrt[n]{a^n + n^k} = a$$

* Број e

$a_n = \left(1 + \frac{1}{n}\right)^n$, $b_n = \left(1 + \frac{1}{n}\right)^{n+1}$

$a_n \uparrow$, $b_n \downarrow$ $a_n < b_n = \left(1 + \frac{1}{n}\right) \cdot a_n$

$a_n < e < b_n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = e = 2{,}7182\ldots$$

① $\forall n \in \mathbb{N}$ $0 < e - \underbrace{\left(1 + \frac{1}{n}\right)^n}_{a_n} < \frac{3}{n}$?

$0 < e - a_n$? $a_n \uparrow$ $\lim_{n\to\infty} a_n = e$ $\Rightarrow a_n < e \Rightarrow e - a_n > 0$

$e - a_n < \frac{3}{n}$? $e - a_n < b_n - a_n = a_n\left(1 + \frac{1}{n}\right) - a_n = \frac{1}{n} \cdot a_n < \frac{e}{n} < \frac{3}{n}$

② $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}\right) = ?$

$a_n = \left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} = b_n$ $/ \ln \uparrow$

$? < \frac{1}{n} < ?$

$\ln\left(1 + \frac{1}{n}\right)^n < 1 < \underbrace{\ln\left(1 + \frac{1}{n}\right)^{n+1}}_{= (n+1)\cdot \ln\left(1 + \frac{1}{n}\right)}$

$\ln\left(1 + \frac{1}{n}\right)^n = n \cdot \ln\left(1 + \frac{1}{n}\right)$

$\Rightarrow \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$ и $\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right)$.

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n} < \ln\left(1+\frac{1}{n-1}\right), \qquad n>1$$

$$\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{1}{n+1}\right)+\dots+\ln\left(1+\frac{1}{2n}\right) < \underbrace{\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{2n}}_{c_n} < \ln\left(1+\frac{1}{n-1}\right)+\ln\left(1+\frac{1}{n}\right)+\dots+\ln\left(1+\frac{1}{2n-1}\right)$$

$$1+\frac{1}{k}=\frac{k+1}{k}$$

$$\ln\left(\frac{\cancel{n+1}}{n}\cdot\frac{\cancel{n+2}}{\cancel{n+1}}\cdot\ldots\cdot\frac{2n+1}{\cancel{2n}}\right) < c_n < \ln\left(\frac{\cancel{n}}{n-1}\cdot\frac{\cancel{n+1}}{\cancel{n}}\cdot\ldots\cdot\frac{2n}{\cancel{2n-1}}\right)$$

$$\ln\left(\frac{2n+1}{n}\right) \xrightarrow{n\to\infty} \ln 2 \qquad\qquad c_n \xrightarrow{\text{T o 2П}} \ln 2 \qquad\qquad \ln\left(\frac{2n}{n-1}\right) \xrightarrow{n\to\infty} \ln 2$$

$$\boxed{\lim_{n\to\infty} \frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{2n} = \ln 2}$$

(3) $\lim_{n\to\infty}\left(\underbrace{1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{n!}}_{c_n}\right) = ?$

$$a_n=\left(1+\frac{1}{n}\right)^n=\sum_{k=0}^{n}\binom{n}{k}\cdot\frac{1}{n^k}$$

$$\binom{n}{k}\cdot\frac{1}{n^k}=\frac{n!}{k!\,(n-k)!}\cdot\frac{1}{n^k}=\frac{1}{k!}\cdot\frac{n!}{(n-k)!\cdot n^k}=$$

$$=\frac{1}{k!}\,\frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{n^k}=$$

$$=\frac{1}{k!}\,\underbrace{\frac{n}{n}}_{\le 1}\cdot\underbrace{\frac{n-1}{n}}_{\le 1}\cdot\underbrace{\frac{n-2}{n}}_{\le 1}\cdots\underbrace{\frac{n-k+1}{n}}_{\le 1} \le \frac{1}{k!}$$

$$\Rightarrow \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \le \sum_{k=0}^{n}\frac{1}{k!}=c_n$$

$$a_n \le c_n \quad \Big/ \lim_{n\to\infty} \quad \Rightarrow \quad e \le \underbrace{\lim_{n\to\infty} c_n}_{\text{ако оно постоји.}}$$

$$b_n=\left(1+\frac{1}{n}\right)^{n+1}=\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{n^k}=\sum_{k=0}^{n+1}\frac{1}{k!}\,\underbrace{\frac{(n+1)\cdot n\cdot(n-1)\cdots(n-k+1)}{n^k}} \;?$$

$$c_n < b_n$$

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{n\cdot(n-1)\cdots(n-1+k)}{n^k}$$

$$= \frac{1}{0!} + \frac{1}{1!}\cdot\frac{n}{n} + \frac{1}{2!}\cdot\frac{n\cdot(n-1)}{n^2} + \dots + \frac{1}{k!}\cdot\frac{n\cdot(n-1)\cdots(n-k+1)}{n^k} + \dots$$

$$\geq 1 + \frac{1}{1!} + \frac{1}{2!}\cdot 1\cdot\frac{n-1}{n} + \dots + \frac{1}{k!}\,\frac{n-1}{n}\cdots\frac{n-k+1}{n} \quad , k<n$$

k фиксирано $\quad n\in\mathbb{N}$

$$\Rightarrow a_n \geq \underbrace{1 + \frac{1}{1!} + \frac{1}{2!}\,\frac{n-1}{n} + \dots + \frac{1}{k!}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n}}_{d_n(k)}$$

$$\lim_{n\to\infty} d_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!} = c_k$$

$$e = \lim_{n\to\infty} a_n \geq \lim_{n\to\infty} d_n^{(k)} = c_k$$

$\Rightarrow$ k фиксирано $\quad c_k \leq e \quad \Rightarrow \quad \lim_{k\to\infty} c_k \leq e$

$$\Rightarrow \boxed{\lim_{n\to\infty} c_n = e}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{n!}\right) = e}$$

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty & \\ 0 & |a|<1 \\ 1 & a=1 \\ \text{не постоји} & \end{cases}$$

$$\frac{\infty}{\infty}$$

1^{∞} не одређен

$$\lim_{n\to\infty} \frac{3^n}{n} \neq \frac{\lim 3^n}{\lim n} = \frac{\infty}{\infty}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \neq \lim\left(\frac{n+1}{n}\right)^{\lim n}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e \qquad \lim\left(\frac{n+1}{n}\right)^{\lim n} = 1^{\infty}$$

0^0

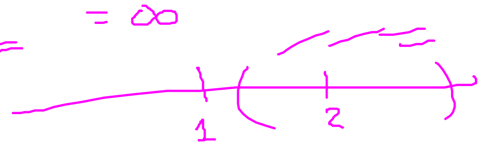
$1^n = 1$ константа

a^n, $a = 1$

$\lim = 1$

$$\lim_{n\to\infty} \left(\frac{2n+1}{n}\right)^n = \text{„} 2^{\infty} \text{“} = \infty$$

$\frac{2n+1}{n} \to 2$



a_n $[a_n]$

$$\lim_{n\to\infty} a_n = a$$

a_n дивергира

$[a_n]$ дивергира ?

$a_n = 1 - \frac{1}{n}$, n парно

$a_n = \frac{1}{n}$, n непарно

$[a_n] = 0$ конвергира

ρ : $(x_1, y_1)\ \rho\ (x_2, y_2)$

$x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 < y_2)$

$(1,1)$ $A = [0,1) \times [0,1)$

$(x,y) \in A$ $x < 1 \Rightarrow (x,y)\ \rho\ (1,1)$

$(1,b)$, $b \in \mathbb{R}$ $(x,y)\ \rho\ (1,b)$

јер $x < 1 \to b$

$\Rightarrow$ не $\mathbb{C} \sim \mathbb{R}^2$