① $\lim_{n\to\infty}\left(\frac{1}{2\cdot4}+\frac{1}{4\cdot6}+\frac{1}{6\cdot8}+\dots+\frac{1}{2n(2n+2)}\right)=$

$\left[\ \frac{1}{2\cdot4}=\frac14\cdot\frac{1}{1\cdot2}=\frac14\cdot\left(\frac11-\frac12\right)\right.$

$\left.\frac{1}{2n(2n+2)}=\frac14\cdot\frac{1}{n\cdot(n+1)}=\frac14\cdot\left(\frac1n-\frac1{n+1}\right)\ \right]$

$=\lim_{n\to\infty}\frac14\left(\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\dots\frac{1}{n\cdot(n+1)}\right)=$

$=\lim_{n\to\infty}\frac14\cdot\left(\frac11-\cancel{\frac12}+\cancel{\frac12}-\cancel{\frac13}+\dots+\cancel{\frac{1}{n-1}}-\cancel{\frac1n}+\cancel{\frac1n}-\underline{\underline{\frac{1}{n+1}}}\right)$

$=\lim_{n\to\infty}\frac14\cdot\left(1-\underbrace{\frac{1}{n+1}}_{0}\right)=\lim_{n\to\infty}\frac14-\frac14\lim_{n\to\infty}\frac{1}{n+1}=\frac14-0=\frac14.$

$$\lim_{n\to\infty}\frac{P_k(n)}{Q_m(n)}=\begin{cases}\frac{a_k}{b_m} & k=m\\ 0 & k<m\\ +\infty & k>m,\ a_k\cdot b_m>0\\ -\infty & k>m,\ a_k\cdot b_m<0\end{cases}$$

$\lim_{n\to\infty}a_n=a \iff \forall\varepsilon>0\ \exists n_0\in\mathbb{N}\ \forall n\geq n_0\ \ |a_n-a|<\varepsilon.$

② a_n, $\lim_{n\to\infty}a_n=0$ $\quad\Rightarrow\quad \lim_{n\to\infty}a_n\cdot b_n=0.$

b_n ограничен низ

? $\forall\varepsilon>0\ \ \exists n_0\in\mathbb{N}\ \ \forall n\geq n_0\ \ |a_n\cdot b_n-0|<\varepsilon$?

$|a_n b_n|<\varepsilon$?

$\varepsilon>0$ произвољно

$n_0=?\ \ \forall n\geq n_0\ \ |a_n\cdot b_n|<\varepsilon$

$\lim_{n\to\infty}a_n=0 \Rightarrow \forall\varepsilon_1>0\ \exists n_1\in\mathbb{N}\ \forall n\geq n_1\ \ |a_n|<\varepsilon_1$

b_n ограничен $\Rightarrow \exists M>0\ \forall n\in\mathbb{N}\ |b_n|\leq M$

$\Rightarrow \forall\varepsilon_1>0\ \exists n_1\in\mathbb{N}\ \forall n\geq n_1\ \ |a_n\cdot b_n|<M\cdot\varepsilon_1=\varepsilon$

$n_0=n_1$ за $\varepsilon_1=\frac{\varepsilon}{M}$ $\underset{n\geq n_0}{\Rightarrow}$ $|a_n\cdot b_n|<M\cdot\varepsilon_1=\varepsilon.$

③ $\lim_{n\to\infty}a_n=a\in\mathbb{R}$. Доказати да $(-1)^n a_n$ конвергира акко је $a=0$.

Да ли $[a_n]$ конвергира?

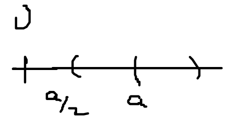
$a=0$

$b_n=(-1)^n$ ограничен $\Big\}\overset{②}{\Rightarrow}$ $(-1)^n a_n$ конвергира $\lim_{n\to\infty}(-1)^n a_n=0$

$(-1)^n a_n$ конвергира, $\lim\limits_{n\to\infty} (-1)^n a_n = b \in \mathbb{R}$

$\forall \varepsilon_1 > 0\ \exists n_1 \in \mathbb{N}\ \forall n \geq n_1 \quad |(-1)^n a_n - b| < \varepsilon_1$

a_n конвергира $\Rightarrow \forall \varepsilon_2 > 0\ \exists n_2 \in \mathbb{N}\ \forall n \geq n_2 \quad |a_n - a| < \varepsilon_2$



ако је $a \neq 0 \Rightarrow \varepsilon_2 = a/2 > 0\ \exists n_2 \in \mathbb{N}\ \forall n \geq n_2 \quad |a_n - a| < a/2$

WLOG $a > 0$

$a_n \in (a/2, \frac{3}{2}a) \ni 0$

n-парне $(-1)^n a_n \in (a/2, 3/2\, a)$

n-непарне $(-1)^n a_n \in (-3/2\, a, -a/2)$ $\subset (0,+\infty)$ $\subset(-\infty,0)$

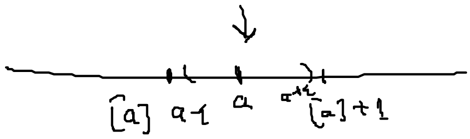
$n \geq n_1 \Rightarrow (-1)^n a_n \in (b - \varepsilon_1, b + \varepsilon_1) \subseteq \underbrace{(\frac{a}{2}, \frac{3}{2}a) \cap (-\frac{3}{2}a, -\frac{a}{2})}_{=\emptyset}$ ↯

$\Rightarrow \underline{\underline{a = 0}}$

$\lim\limits_{n\to\infty} a_n = a$

$\lim\limits_{n\to\infty} [a_n] = ?$

$a \in \mathbb{R} \setminus \mathbb{Z}$



$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 \quad |a_n - a| < \varepsilon$

$a_n \in (a - \varepsilon, a + \varepsilon)$

$\varepsilon = \min\{a - [a], [a] + 1 - a\}$

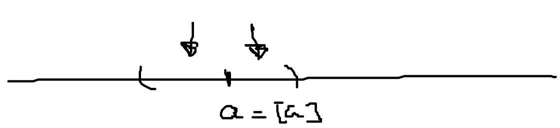
$\forall n \geq n_0 \quad a_n \in (a - \min, a + \min) \Rightarrow [a_n] = [a]$

$\subseteq ([a], [a]+1)$

$\Rightarrow [a_n]$ конвергира јер је почев од неког константан

и $\lim\limits_{n\to\infty} [a_n] = [a]$.

$a \in \mathbb{Z} \quad \forall n \geq n_0 \quad a_n \in (a - \varepsilon, a + \varepsilon)$



ако $\exists n_0 \in \mathbb{N}\ \forall n \geq n_0 \quad a_n \in [a, a + \varepsilon)$

онда $[a_n] = [a] = a$ и низ конв.

или ако $\exists n_0 \in \mathbb{N}\ \forall n \geq n_0 \quad a_n \in (a - \varepsilon, a)$ онда $[a_n] = [a] - 1 = a - 1$ и тада низ конв.

$\forall n \in \mathbb{N}\ \exists n_1 \in \mathbb{N}$ важи $a_n < a \leq a_{n_1}$ или $a_{n_1} < a \leq a_n$

онда $[a_{n_1}] \neq [a_n]$ и низ не конв. јер

T. $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$, $a < b \Rightarrow \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 \quad a_n < b_n$

T. $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$, $\exists n_0 \in \mathbb{N}\ \forall n \geq n_0 \quad a_n < b_n \Rightarrow a \leq b$

Т о 2 полицајца (То2П): $(a_n), (b_n), (c_n)$, $\exists n_0 \in \mathbb{N}$ $\forall n \geq n_0$

$$a_n \leq b_n \leq c_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = b \in \mathbb{R}$$

$$\Rightarrow \underline{\underline{\lim_{n\to\infty} b_n = b}}$$

(4) $$\lim_{n\to\infty} \Big(\underbrace{\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}}_{n} \Big) = ?$$

$$a_n = n \cdot \frac{1}{\sqrt{n^2+n}}$$

$$b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$$

$$c_n = n \cdot \frac{1}{\sqrt{n^2+1}}$$

$$\underline{\underline{a_n \leq b_n \leq c_n}} \qquad \frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+k}} \leq \frac{1}{\sqrt{n^2+1}} \qquad 1 \leq k \leq n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\underset{\downarrow 0}{1/n}}} = \frac{1}{\sqrt{1+0}} = 1$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} = 1$$

$$\overset{\text{То2П}}{\Longrightarrow} \lim_{n\to\infty} b_n = 1.$$

(5) $$\lim_{n\to\infty} \underbrace{\sqrt[n]{\pi^n + 2021^n + (7e)^{2n}}}_{b_n} = ?$$

π^n

2021^n

$(7e)^{2n} = (49 \cdot e^2)^n$

$$a_n = 2021$$

$$c_n = \sqrt[n]{2021^n \cdot 3} = 2021 \cdot \underline{\underline{\sqrt[n]{3}}}$$

$$\left.\begin{aligned} &\lim_{n\to\infty} a_n = 2021 \ \checkmark \\ &\lim_{n\to\infty} c_n = 2021 \cdot \underbrace{\lim_{n\to\infty} \sqrt[n]{3}}_{=1\ ?} = 2021 \end{aligned}\right\} \Rightarrow \underline{\underline{\lim_{n\to\infty} b_n = 2021}}$$

$$* \ \boxed{\lim_{n\to\infty} \sqrt[n]{a} = 1 \quad \text{за } a > 0}$$

$$a = 1 \qquad \sqrt[n]{a} = 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{1} = 1 \ \checkmark$$

$a>1 \quad \sqrt[n]{a} > 1 \quad \Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{a} \geq 1$

$b_n = \sqrt[n]{a} - 1 > 0 \qquad ? \lim_{n\to\infty} b_n = 0 ?$

$1 + b_n = \sqrt[n]{a} \;/{}^{n} \Rightarrow \quad a = (1+b_n)^n \geq 1 + n \cdot b_n \quad \Rightarrow b_n \leq \frac{a-1}{n}$

Бернулијева
неједнакост
$(1+c)^n > 1+nc$

$a_n = 0$
$c_n = \frac{a-1}{n}$

$0 \leq b_n \leq \frac{a-1}{n} \overset{\text{Тo2П}}{\Rightarrow} \lim_{n\to\infty} b_n = 0.$

$\downarrow n\to\infty$
0

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1 \quad , a>1$

$a<1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt[n]{a}}} =$

$= \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{a}}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{a}}} = \frac{1}{1} = 1$

$\frac{1}{a} > 1$

(6) $\lim_{n\to\infty} \underbrace{\sqrt[n]{a_1^n + a_2^n + \dots + a_k^n}}_{b_n} = \max\{a_1, \dots, a_k\} \quad , \quad a_1, \dots, a_k \geq 0$

$a_n = \max\{a_1, \dots, a_k\} \leq \sqrt[n]{a_1^n + \dots + a_k^n}$

$c_n = \sqrt[n]{k \cdot \max\{a_1, \dots, a_k\}^n} \geq \sqrt[n]{a_1^n + \dots + a_k^n}$

$\lim_{n\to\infty} a_n = \max\{a_1, \dots, a_k\}$

$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \max\{a_1, \dots, a_k\} \cdot \underset{\substack{\downarrow \\ 1}}{\sqrt[n]{k}} = \max\{a_1, \dots, a_k\}$

То2П
$\Rightarrow \lim_{n\to\infty} b_n = \max\{a_1, \dots, a_k\}$

(7) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

$\lim_{n\to\infty} (\underbrace{\sqrt[n]{n} - 1}_{b_n}) = 0 \; ?$

$n = (b_n+1)^n \geq 1 + n b_n \quad \Rightarrow \quad b_n \leq \frac{n-1}{n} \to 1 \quad \Rightarrow$ није добра оцена

$n = (b_n+1)^n = \binom{n}{0} b_n^0 \cdot 1^n + \binom{n}{1} b_n^1 \cdot 1^{n-1} + \dots + \binom{n}{n-1} b_n^{n-1} \cdot 1 + \binom{n}{n} b_n^n \cdot 1$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$= \underbrace{\frac{n!}{0!\cdot n!}}_{1} + \frac{n!}{1!\cdot(n-1)!}\, b_n + \frac{n!}{2!\cdot(n-2)!}\cdot b_n^2 + \dots + b_n^n$$

$$= 1 + n\cdot b_n + \frac{n\cdot(n-1)}{2} b_n^2 + \dots \geq \frac{n\cdot(n-1)}{2}\cdot b_n^2$$

$$b_n^2\cdot\frac{n(n-1)}{2} \leq n \Rightarrow b_n^2 \leq \frac{2}{n-1}$$

$$0 \leq b_n \leq \sqrt{\frac{2}{n-1}} = c_n \xrightarrow{n\to\infty} 0$$

Тогда $\Rightarrow$ $\lim\limits_{n\to\infty} b_n = 0 \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$

$$\boxed{\lim_{n\to\infty} \sqrt[n]{n} = 1}$$

(*) $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\cdot b^{n-k}$$

ПМИ: (БИ) $n=1$ $\quad (a+b)^1 = a+b = \underbrace{\binom{1}{0}}_{1}\cdot a^0\cdot b^1 + \underbrace{\binom{1}{1}}_{1}\cdot a^1\cdot b^0$

(ИК) $\quad (a+b)^n = \sum\limits_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$n+1$: $$(a+b)^{n+1} = (a+b)\cdot\sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k} a^{k+1}\cdot b^{n-k} + \sum_{k=0}^{n}\binom{n}{k} a^k\cdot b^{n-k+1}$$

$l = k+1$

$$= \sum_{l=1}^{n+1}\binom{n}{l-1} a^l\cdot b^{n-l+1} + \sum_{k=0}^{n}\binom{n}{k} a^k b^{n+1-k}$$

$$= \sum_{k=1}^{n}\left(\binom{n}{k-1}+\binom{n}{k}\right)\cdot a^k\cdot b^{n-k+1} + \underbrace{\binom{n}{n+1-1}}_{=n}\cdot a^{n+1}\cdot b^{n-(n+1)-1} + \binom{n}{0}\cdot a^0\cdot b^{n+1}$$

$$= \underbrace{1}_{\binom{n+1}{n+1}}\, a^{n+1} + \underbrace{1}_{\binom{n+1}{0}}\cdot b^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k} a^k b^{n+1-k} = \sum_{k=0}^{n+1}\binom{n+1}{k} a^k b^{n+1-k}.$$

$$\binom{n}{k-1}+\binom{n}{k} = \frac{n!}{(k-1)!\,(n+1-k)!} + \frac{n!}{k!\cdot(n-k)!}$$

$$= \frac{n!}{(k-1)!\cdot(n-k)!}\left(\frac{1}{n+1-k}+\frac{1}{k}\right)$$

$$= \frac{n!\,(n+1)}{k!\,(n+1-k)!} = \binom{n+1}{k}$$

8) a) $\lim_{n\to\infty} b_n = ?$

$$8b_{n+2} = 6b_{n+1} - b_n \quad , \quad b_0 = 8 \quad , \quad b_1 = \frac{13}{4}$$

? $\lim_{n\to\infty} b_n = ?$ пп. $b = \lim_{n\to\infty} b_n$

$$\Rightarrow 8 \cdot \underbrace{\lim_{n\to\infty} b_{n+2}}_{b} = 6 \underbrace{\lim_{n\to\infty} b_{n+1}}_{b} - \underbrace{\lim_{n\to\infty} b_n}_{b}$$

$$8b = 5b \Rightarrow \underline{\underline{b=0}}.$$

И сада још треба показати да b_n конвергира.

$b_n = ?$ $\quad 8x^2 = 6x - 1 \quad 8x^2 - 6x + 1 = 0 \Rightarrow x_{1/2} = \frac{6 \pm \sqrt{36-32} = 4}{16}$

$$x_1 = \frac{1}{4}$$

$$x_2 = \frac{1}{2}$$

$$b_n = c_1 \cdot x_1^n + c_2 \cdot x_2^n = \frac{c_1}{4^n} + \frac{c_2}{2^n}$$

$$\lim_{n\to\infty} b_n = c_1 \cdot \lim_{n\to\infty} \frac{1}{4^n} + c_2 \lim_{n\to\infty} \frac{1}{2^n} = 0$$

б) $\lim_{n\to\infty} \sqrt[n]{b_n} = ?$