Граничне вредности низова

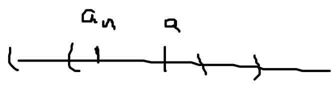
$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

$\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$ уколико $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\quad n \geq n_0 \Rightarrow |a - a_n| < \varepsilon$

и тада кажемо да низ $\{a_n\}$ КОНВЕРГИРА.

Ако низ не конвергира онда кажемо да ДИВЕРГИРА.

$$\left.\begin{array}{ll} \lim\limits_{n\to\infty} a_n = +\infty & \forall M \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\quad n \geq n_0 \Rightarrow a_n > M \\ \lim\limits_{n\to\infty} a_n = -\infty & \forall m \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\quad n \geq n_0 \Rightarrow a_n < m \end{array}\right\} a_n \text{ ДИВЕРГИРА}$$



$a_n = (-1)^n$ уколико $\lim\limits_{n\to\infty} a_n = a$

$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\quad n \geq n_0\quad |a_n - a| < \varepsilon \Rightarrow n \geq n_0\quad a_n \in (a - \varepsilon, a + \varepsilon)$

$\varepsilon = 1/2 \Rightarrow (-1)^n \in (a - 1/2, a + 1/2) \Rightarrow a \in ((-1)^n - 1/2, (-1)^n + 1/2)$

n парно $\Rightarrow a \in (\frac{1}{2}, 3/2)$ $\qquad a \in (-\frac{3}{2}, -\frac{1}{2}) \cap (1/2, 3/2) = \emptyset$

n непарно $\Rightarrow a \in (-3/2, -1/2)$

$\Rightarrow$ овакво a не постоји!

$\Rightarrow (-1)^n$ ДИВЕРГИРА и НЕМА ГРАНИЧНУ ВРЕДНОСТ.

$a, b \in \overline{\mathbb{R}}$

$$a + b = \begin{cases} a + b, & a, b \in \mathbb{R} \\ +\infty, & a = +\infty,\ b \in \mathbb{R} \cup \{+\infty\} \\ -\infty, & a = -\infty,\ b \in \mathbb{R} \cup \{-\infty\} \\ \text{није дефинисано}, & a = +\infty,\ b = -\infty \end{cases}$$

$$a \cdot b = \begin{cases} a \cdot b, & a, b \in \mathbb{R} \\ +\infty, & a = +\infty,\ b > 0;\ a = -\infty, b < 0 \\ -\infty, & a = -\infty,\ b > 0;\ a = +\infty, b < 0 \\ \text{није дефинисано}, & a = \pm\infty,\ b = 0. \end{cases}$$

a_n, b_n : $\lim\limits_{n\to\infty} a_n = a \in \overline{\mathbb{R}}$, $\lim\limits_{n\to\infty} b_n = b \in \overline{\mathbb{R}}$

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} (a_n + b_n) := \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = a + b \\ \lim\limits_{n\to\infty} (a_n \cdot b_n) := \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = a \cdot b \end{array}\right\} \text{дефинисано када је дефинисана десна страна}$$

$$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{a}{b}, \quad b_n \neq 0,\ \lim\limits_{n\to\infty} b_n \neq 0 \qquad \left(\frac{1}{\pm\infty} := 0\right)$$

$a_n = c$ за $n \geq n_0$

$\Rightarrow \boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} c = c}$

1) $$\boxed{\lim_{n\to\infty} a^n = \begin{cases} 0, & |a|<1 \\ 1, & a=1 \\ +\infty, & a>1 \\ \text{не постоји}, & a \leq -1 \end{cases}}$$

1° $a=1$ $\quad 1^n = 1 \quad \Rightarrow \lim_{n\to\infty} 1^n = 1.$

2° $|a|<1$ $\quad ?\ \lim_{n\to\infty} a^n = 0\ ?$ $\quad \checkmark$

$?\ \forall \varepsilon>0\ \exists n_0\in\mathbb{N}\ \forall n\geq n_0\ |a^n - 0| < \varepsilon\ ?$

$\varepsilon>0$ произвољно $\quad n_0 = ?$

$n \geq n_0 \Rightarrow \underset{|a|^n}{\underset{\shortparallel}{|a^n|}} < \varepsilon \quad / \log_{|a|} \downarrow, \quad |a|<1$

$f\downarrow \quad x \leq y$

$f(x) \geq f(y)$

$\log_{|a|}|a|^n > \log_{|a|}\varepsilon$

$n > \log_{|a|}\varepsilon \quad \Rightarrow n_0 = [\log_{|a|}\varepsilon] + 1.$

3° $a>1$ $\quad \lim_{n\to\infty} a^n = +\infty\ ?$

$\forall M>1\ \exists n_0\in\mathbb{N}\ \forall n\geq n_0\ a^n > M\ ?$

$M>1$ произвољно $\quad n_0 = ?$

$n\geq n_0 \quad a^n > M \quad / \log_a \uparrow \quad a>1$

$n > \log_a M$

$n_0 = [\log_a M] + 1$

4° $a \leq -1$ $\quad ?\ \lim_{n\to\infty} a^n$ не постоји ?

Претпоставимо супротно, $\lim_{n\to\infty} a^n = b \in \overline{\mathbb{R}}$.

1° $b\in\mathbb{R}$ $\quad \forall\varepsilon>0\ \exists n_0\in\mathbb{N}\ \forall n\geq n_0\ |a^n - b| < \varepsilon \Rightarrow b\in(a^n-\varepsilon, a^n+\varepsilon)\ \forall n\geq n_0$

$\varepsilon = 1 \Rightarrow \exists n_0\in\mathbb{N}\ \forall n\geq n_0\ b\in(a^n-\varepsilon, a^n+\varepsilon)$

n парно $\quad b\in(\underset{\underset{1}{\rotatebox{90}{$\leq$}}}{|a|^n}-1, |a|^n+1) \subseteq [0,+\infty)$

n непарно $\quad b\in(-|a|^n-1, \underbrace{-|a|^n+1}_{\leq 0}) \subseteq (-\infty, 0)$

$\Rightarrow b\in(0,+\infty)\cap(-\infty,0) = \emptyset$ ↯

2° $b = +\infty$ $\quad \forall M \in \mathbb{R} \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \quad a^n > M$

$M = 0 \quad \Rightarrow \quad n$ - непарно $\geq n_0 \quad a^n < 0$ ↯

$b = -\infty \quad n$ - парно $\geq n_0 \quad \underline{\underline{a^n > 0}}$ ↯

$\Rightarrow \lim\limits_{n\to\infty} a^n$ не постоји за $a \leq -1$.

(2) $p_k(x) = a_k \cdot x^k + a_{k-1} x^{k-1} + \dots + a_0 \quad , \quad a_k \neq 0$

$q_m(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0 \qquad b_m \neq 0$

$$\boxed{\lim_{n\to\infty} \frac{p_k(n)}{q_m(n)} = \begin{cases} \dfrac{a_k}{b_m} & , \ k = m \\ 0 & , \ k < m \\ +\infty & , \ k > m \quad a_k \cdot b_m > 0 \\ -\infty & , \ k > m \quad a_k \cdot b_m < 0 \end{cases}}$$

$$\lim_{n\to\infty} \frac{a_k \cdot n^k + a_{k-1} \cdot n^{k-1} + \dots + a_1 n + a_0}{b_m \cdot n^m + b_{m-1} \cdot n^{m-1} + \dots + b_1 n + b_0} = \lim_{n\to\infty} \frac{n^k}{n^m} \cdot \frac{a_k + a_{k-1} \frac{1}{n} + \dots + a_1 \cdot \frac{1}{n^{k-1}} + \frac{a_0}{n^k}}{b_m + b_{m-1} \cdot \frac{1}{n} + \dots + b_1 \cdot \frac{1}{n^{m-1}} + \frac{b_0}{n^m}}$$

$$\underset{(*)}{=} \lim_{n\to\infty} n^{k-m} \cdot \lim_{n\to\infty} \frac{\dots}{\dots} = ☆$$

уколико је дефинисан производ ова два lim ...

$$\lim_{n\to\infty} n^l = \begin{cases} 0 & l < 0 \\ 1 & l = 0 \\ +\infty & l > 0 \end{cases}$$

Г 1° $l < 0$:

$\lim\limits_{n\to\infty} n^l = 0$? $\qquad \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ n \geq n_0 \quad n^l < \varepsilon$

$m = -l > 0 \qquad n^l = \dfrac{1}{n^m} < \varepsilon$

$n^m > \dfrac{1}{\varepsilon} \quad \Rightarrow \quad n > \dfrac{1}{\varepsilon^{1/m}}$

$n_0 = \left[\dfrac{1}{\varepsilon^{1/m}}\right] + 1 \quad \Rightarrow \quad n \geq n_0 \geq \dfrac{1}{\varepsilon^{1/m}} \quad \Rightarrow \quad \dfrac{1}{n^m} < \varepsilon$ ✓

2° $l = 0 \quad \Rightarrow n^l = 1 \quad \Rightarrow \lim\limits_{n\to\infty} 1 = 1$

3° $l > 0 \quad \lim\limits_{n\to\infty} n^l = +\infty \qquad \forall M > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \quad n^l > M$

$n > M^{1/l}$

$n_0 = [M^{1/l}] + 1$ ┘

$$\star = \lim_{n\to\infty} n^{k-m} \cdot \lim_{n\to\infty} \frac{a_k + a_{k-1}\cdot\frac{1}{n} + a_{k-2}\cdot\frac{1}{n^2} + \dots + a_1\frac{1}{n^{k-1}} + a_0\cdot\frac{1}{n^k}}{b_m + b_{m-1}\cdot\frac{1}{n} + b_{m-2}\cdot\frac{1}{n^2} + \dots + b_0\cdot\frac{1}{n^m}}$$

$$\underset{(\star)}{=} \lim_{n\to\infty} n^{k-m} \cdot \frac{\overbrace{\lim_{n\to\infty}\left(a_k + a_{k-1}\cdot\frac{1}{n} + \dots + a_0\cdot\frac{1}{n^k}\right)}^{k \text{ сабирака}}}{\lim_{n\to\infty}\left(b_m + b_{m-1}\cdot\frac{1}{n} + \dots + b_0\cdot\frac{1}{n^m}\right)}$$

(⋆) ⇓ Уколико постоји десна страна

$$\underset{(\star)}{=} \lim_{n\to\infty} n^{k-m} \cdot \frac{\lim_{n\to\infty} a_k + \underbrace{\lim_{n\to\infty} a_{k-1}\cdot\frac{1}{n}}_{=0} + \dots + \underbrace{\lim_{n\to\infty} a_0\cdot\frac{1}{n^k}}_{=0}}{\lim_{n\to\infty} b_m + \underbrace{\lim_{n\to\infty} b_{m-1}\cdot\frac{1}{n}}_{=0} + \dots + b_0\cdot\underbrace{\lim_{n\to\infty}\frac{1}{n^m}}_{=0}} = \lim_{n\to\infty} n^{k-m}\cdot\frac{a_k}{b_m}$$

$$= \begin{cases} 0 & k<m \\ \frac{a_k}{b_m} & k=m \\ \frac{a_k}{b_m}\cdot(+\infty) & k>m \end{cases}$$

(3) $$\lim_{n\to\infty}\frac{1^3+2^3+\dots+n^3}{n^4} = \lim_{n\to\infty}\underbrace{\left(\frac{1^3}{n^4}+\frac{2^3}{n^4}+\dots+\frac{n^3}{n^4}\right)}_{n \text{ сабирака}} = \text{не писати, није добро!}$$

$$\left(= \lim_{n\to\infty}\frac{1^3}{n^4} + \lim_{n\to\infty}\frac{2^3}{n^4} + \dots + \lim_{n\to\infty}\frac{n^3}{n^4} = 0+0+0+\dots+0 = 0 \right)$$

немамо фиксиран број сабирака и не можемо проћи лимесом кроз збир!

$$1^3+2^3+\dots+n^3 = \left(\frac{n\cdot(n+1)}{2}\right)^2$$ → ово је био задатак кад смо радили ПМИ

$$\Rightarrow \lim_{n\to\infty}\frac{1^3+2^3+\dots+n^3}{n^4} = \lim_{n\to\infty}\frac{\left(\frac{n\cdot(n+1)}{2}\right)^2}{n^4} = \frac{1}{4}$$

(4) $$\lim_{n\to\infty}\frac{\binom{n}{k}}{n^k} = ? \qquad \binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}, \quad n! = n\cdot(n-1)\cdots 1$$

$$\lim_{n\to\infty}\frac{\binom{n}{k}}{n^k} = \lim_{n\to\infty}\frac{n!}{k!\,(n-k)!\cdot n^k} = \frac{1}{k!}\lim_{n\to\infty}\frac{\overbrace{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}^{k\text{-чиниоца}}}{n^k = \underbrace{n\cdot n\cdots n}_{k}}$$

$$= \frac{1}{k!}\lim_{n\to\infty}\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n} = \frac{1}{k!}\underbrace{\lim_{n\to\infty}\frac{n}{n}}_{=1}\;\underbrace{\lim_{n\to\infty}\frac{n-1}{n}}_{=1}\cdots\underbrace{\lim_{n\to\infty}\frac{n-k+1}{n}}_{=1} = \frac{1}{k!}$$

(5) $\lim\limits_{n\to\infty}\left(\frac{1}{2\cdot 4}+\frac{1}{4\cdot 6}+\frac{1}{6\cdot 8}+\dots+\frac{1}{2n\cdot(2n+2)}\right)=$

$$=\lim_{n\to\infty}\frac{1}{4}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n\cdot(n+1)}\right)$$

$$\left\ulcorner \frac{1}{k\cdot(k+1)}=\frac{1}{k}-\frac{1}{k+1}\right\lrcorner$$

$$=\lim_{n\to\infty}\frac{1}{4}\cdot\left(\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\dots}+\cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n}}+\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$=\lim_{n\to\infty}\frac{1}{4}\cdot\left(1-\frac{1}{n+1}\right)=\frac{1}{4}-\frac{1}{4}\underbrace{\lim_{n\to\infty}\frac{1}{n+1}}_{=0}=\frac{1}{4}.$$