* Диференцни рачун

· a_0, a_1 – знано

$a_{n+2} = p a_{n+1} + q a_n$, $p, q \in \mathbb{R}$ → диференцна ј-на 2. реда

· Наћи вредности a_n у зависности од n, а не од a_{n-1} и a_{n-2}.

Посматрамо карактеристичну ј-ну:

$$x^2 = px + q$$

$$x^2 - px - q = 0 \rightarrow x_1, x_2 \text{ решења}$$

1° $x_1 \neq x_2$, $x_1, x_2 \in \mathbb{C}$

$$a_n = c_1 x_1^n + c_2 \cdot x_2^n \quad , c_1, c_2 \in \mathbb{R}$$

2° $x_1 = x_2$

$$a_n = c_1 x_1^n + c_2 n \cdot x_1^n \quad , c_1, c_2 \in \mathbb{R}$$

Да ли је a_n заиста решење диференцне ј-не?

$\left.\begin{matrix} a_0 \\ a_1 \end{matrix}\right\}$ дати → хоћемо да и они задовољавају $a_n = \dots$

нпр. $x_1 \neq x_2$

$$\left.\begin{aligned} a_0 &= c_1 \cdot x_1^0 + c_2 \cdot x_2^0 = c_1 + c_2 \\ a_1 &= c_1 \cdot x_1 + c_2 x_2 \end{aligned}\right\} \text{наћемо } c_1, c_2$$

? $a_{n+2} = p \cdot a_{n+1} + q \cdot a_n$?

$$c_1 \cdot x_1^{n+2} + c_2 \cdot x_2^{n+2} = p \cdot (c_1 x_1^{n+1} + c_2 \cdot x_2^{n+1}) + q(c_1 \cdot x_1^n + c_2 \cdot x_2^n)$$

$$c_1 x_1^n \underbrace{(x_1^2 - p x_1 - q)}_{=0} + c_2 \cdot x_2^n \underbrace{(x_2^2 - p x_2 - q)}_{=0} = 0 \quad \checkmark$$

① $a_{n+2} = 5 a_{n+1} - 6 a_n$

$a_1 = 5$, $a_2 = 13$

карактер. ј-на $x^2 = 5x - 6 \rightarrow x^2 - 5x + 6 = 0$

$$x_{1/2} = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$x_1 = 2$

$x_2 = 3$

$x_1 \neq x_2$

$$a_n = c_1 \cdot 2^n + c_2 \cdot 3^n$$

$a_1 = 5 \Rightarrow c_1 \cdot 2 + c_2 \cdot 3 = 5 \quad / \cdot 2$

$a_2 = 13 \Rightarrow c_1 \cdot 4 + c_2 9 = 13$

$2c_1 + 3c_2 = 5 \Rightarrow \boxed{c_1 = 1}$

$3c_2 = 3 \Rightarrow \boxed{c_2 = 1}$

Да ли је $a_n = 2^n + 3^n$ решење диф. ј-не ?

(Би) $a_1 = 2^1 + 3^1$ ✓

$a_2 = 2^2 + 3^2$ ✓

(ИК) Претпоставимо да је за неко $n \in \mathbb{N}$ $a_n = 2^n + 3^n$ и $a_{n+1} = 2^{n+1} + 3^{n+1}$.

Да ли је $a_{n+2} = 2^{n+2} + 3^{n+2}$?

$$a_{n+2} = 5a_{n+1} - 6a_n = 5(2^{n+1} + 3^{n+1}) - 6(2^n + 3^n)$$

$$= 2^n(5 \cdot 2 - 6) + 3^n(3 \cdot 5 - 6) =$$

$$= 2^n \cdot 4 + 3^n \cdot 9 = 2^{n+2} + 3^{n+2} \quad ✓✓.$$

2) $a_{n+2} = 2a_{n+1} - a_n$

$a_1 = 4,\ a_2 = 12$

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каракт. ј-на : $x^2 = 2x - 1 \rightsquigarrow x^2 - 2x + 1 = 0 \rightarrow x_1 = x_2 = 1$

$a_n = c_1 \cdot 1^n + nc_2 1^n = c_1 + nc_2$?

$c_1, c_2 = ?$ $\left.\begin{array}{l} a_1 = 4 = c_1 + c_2 \\ a_2 = 12 = c_1 + 2c_2 \end{array}\right\} \Rightarrow \begin{array}{l} c_2 = 8 \\ c_1 = -4 \end{array}$

? $a_n = -4 + 8n$?

(Би) ✓

(ИК) $a_n = -4 + 8n$, $a_{n+1} = -4 + 8(n+1) \xRightarrow{?}$? $a_{n+2} = -4 + 8(n+2)$?

$$a_{n+2} = 2a_{n+1} - a_n = -8 + 16(n+1) - (-4 + 8n) = -4 + 8n + 16 = -4 + 8(n+2)$$ ✓

• диференцна ј-на 3. степена /реда

$a_{n+3} = pa_{n+2} + qa_{n+1} + ra_n$, $p, q, r \in \mathbb{R}$

карактеристична ј-на:

$x^3 = px^2 + qx + r \rightsquigarrow x_1, x_2, x_3 \in \mathbb{C}$ решења

1° $x_1 \neq x_2 \neq x_3 \neq x_1$

$a_n = c_1 x_1^n + c_2 x_2^n + c_3 x_3^n$, $c_1, c_2, c_3 \in \mathbb{R}$

2° $x_1 = x_2 \neq x_3$

$a_n = c_1 x_1^n + n \cdot c_2 x_1^n + c_3 \cdot x_3^n$

3° $x_1 = x_2 = x_3$

$$a_n = c_1 x_1^n + n c_2 x_1^n + n^2 c_3 x_1^n$$

(3) $a_{n+3} = -a_{n+2} + 17 a_{n+1} - 15 a_n$

$a_0 = 1,\ a_1 = 3,\ a_2 = 9$

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карактер. j-на: $x^3 = -x^2 + 17x - 15$ $\quad -2-8$

$x^3 + x^2 - 17x + 15 = 0 \longrightarrow x_1 = 1$

$1 + 1 - 17 + 15 = 0$

$x^3 + x^2 - 17x + 15 : (x-1) = \underbrace{x^2 + 2x - 15} \longrightarrow x_{2,3} = \dfrac{-2 \pm \sqrt{4+60}}{2}$

$$\begin{array}{l} x^3 - x^2 \\ \hline 2x^2 - 17x + 15 \\ 2x^2 - 2x \\ \hline -15x + 15 \\ -15x + 15 \\ \hline 0 \end{array}$$

$x_3 = -5$

$x_2 = +3$

$x_1 \neq x_2 \neq x_3 \neq x_1$

$$a_n = c_1 \cdot 1^n + c_2 \cdot (+3)^n + c_3 \cdot (-5)^n$$

$a_0 = 1 = c_1 + c_2 + c_3$

$a_1 = 3 = c_1 + 3c_2 - 5c_3$

$a_2 = 9 = c_1 + 9c_2 + 25c_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 3 & -5 & 3 \\ 1 & 9 & 25 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -6 & 2 \\ 0 & 8 & 24 & 8 \end{array}\right] \cdot -4$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -6 & 2 \\ 0 & 0 & 48 & 0 \end{array}\right] \to \begin{array}{l} c_3 = 0 \\ c_2 = 1 \\ c_1 = 0 \end{array}$$

$a_n = 3^n$

(БИ) ✓

(ИК): $a_n = 3^n,\ a_{n+1} = 3^{n+1},\ a_{n+2} = 3^{n+2}$

$$a_{n+3} = -3^{n+2} + 17\, 3^{n+1} - 15 \cdot 3^n = 3^n(-9 + 17 \cdot 3 - 15) = 3^{n+1}(-3 + 17 - 5)$$

$$= 3^{n+1} \cdot 9 = 3^{n+3} \checkmark$$

(4) $a_{n+1} = 2a_n - b_n \rightsquigarrow b_n = 2a_n - a_{n+1}$

$b_{n+1} = a_n + 4b_n \qquad b_{n+1} = 2a_{n+1} - a_{n+2} = a_n + 4(2a_n - a_{n+1})$

$a_0 = 2, \; b_0 = 1$

$a_n = ?\;, \; b_n = ?$

$a_{n+2} = 2a_{n+1} - a_n - 4(2a_n - a_{n+1})$

$= 6a_{n+1} - 9a_n$

$a_1 = 2a_0 - b_0 = 4 - 1 = 3$

$b_1 = a_0 + 4b_0 = 6$

$\Rightarrow a_{n+2} = 6a_{n+1} - 9a_n$

$x^2 = 6x - 9 \rightsquigarrow x^2 - 6x + 9 = 0 \Rightarrow x_1 = x_2 = 3$

$a_n = c_1 \cdot 3^n + n c_2 3^n$

$a_0 = 2 = c_1$

$a_1 = 3 = 3 \cdot c_1 + 3c_2 = 6 + 3c_2 \Rightarrow c_2 = -1$

$a_n = 2 \cdot 3^n - n \cdot 3^n$?

(БИ) ✓

(ИК) $a_n = 2 \cdot 3^n - n \cdot 3^n \;, \; a_{n+1} = 2 \cdot 3^{n+1} - (n+1) \cdot 3^{n+1}$

$a_{n+2} = 6 \cdot (2 \cdot 3^{n+1} - (n+1) \cdot 3^{n+1}) - 9(2 \cdot 3^n - n 3^n)$

$= 4 \cdot 3^{n+2} - 2 \cdot (n+1) \cdot 3^{n+2} - 2 \cdot 3^{n+2} + n \cdot 3^{n+2}$

$= 2 \cdot 3^{n+2} - (n+2) \cdot 3^{n+2}$ ✓

$b_n = ?$

$a_n = b_{n+1} - 4b_n$

$a_{n+1} = 2a_n - b_n \Rightarrow b_{n+2} - 4b_{n+1} = 2b_{n+1} - 8b_n - b_n$

$b_{n+2} = 6b_{n+1} - 9b_n$

$b_n = d_1 \cdot 3^n + n \cdot d_2 \cdot 3^n \;, \; b_0 = 1, \; b_1 = 6$

$d_1 = 1$

$3\underbrace{d_1}_{=1} + 3\underbrace{d_2}_{=1} = 6 \Rightarrow d_2 = 1$

$b_n = 3^n + n \cdot 3^n$

...

- $a_{n+1} = \dfrac{p a_n + q}{r \cdot a_n + s}$ $\quad p, q, r, s \in \mathbb{R}$

a_1 - дато

$a_n = ?$ $\rightsquigarrow$ $a_n = \dfrac{x_n}{y_n}$, $y_n \neq 0 \Rightarrow \dfrac{x_{n+1}}{y_{n+1}} = \dfrac{p \cdot \frac{x_n}{y_n} + q}{r \cdot \frac{x_n}{y_n} + s} \cdot \dfrac{y_n}{y_n} = \dfrac{p \cdot x_n + q y_n}{r x_n + s y_n}$

x_n, y_n : $\quad x_{n+1} = p \cdot x_n + q y_n$ $\qquad \dfrac{x_1}{y_1} = a_1 \Rightarrow \begin{matrix} x_1 = a_1 \\ y_1 = 1 \end{matrix}$

$y_{n+1} = r \cdot x_n + s y_n$

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(5) $a_{n+1} = \dfrac{1 - 4a_n}{1 - 6a_n}$, $a_1 = 1$

$a_n = \dfrac{x_n}{y_n}$; $\quad x_{n+1} = -4x_n + y_n$ $\qquad x_1 = y_1 = 1$ $\rightarrow$ можемо узети шта год само да $\dfrac{x_1}{y_1} = a_1$.

$y_{n+1} = -6x_n + y_n$

$y_n = x_{n+1} + 4x_n$

$x_{n+2} + 4x_{n+1} = -6x_n + x_{n+1} + 4x_n$

$x_{n+2} = -3x_{n+1} - 2x_n$

$\rightsquigarrow x^2 = -3x - 2 \Rightarrow \underbrace{x^2 + 3x + 2}_{(x+1)(x+2)} = 0 \Rightarrow \begin{matrix} x_1 = -1 \\ x_2 = -2 \end{matrix}$

$x_1 \neq x_2$

$x_n = c_1 \cdot (-1)^n + c_2 \cdot (-2)^n$

$x_1 = 1$, $x_2 = -3$

$1 = c_1 \cdot (-1)^1 + c_2 \cdot (-2)^1 = -c_1 - 2c_2$

$-3 = c_1 \cdot 1 + c_2 \cdot 4$ $\qquad > \quad 2c_2 = -2$ $\quad \boxed{\begin{matrix} c_2 = -1 \\ c_1 = 1 \end{matrix}}$

$x_n = (-1)^n - (-2)^n$

...

индукција за вежбу

$y_n = ?$: $\quad x_n = \dfrac{y_n - y_{n+1}}{6}$ $\Bigg\} \Rightarrow \dfrac{y_{n+1} - y_{n+2}}{6} = -4 \cdot \dfrac{y_n - y_{n+1}}{6} + y_n \;/\; 6$

$x_{n+1} = -4x_n + y_n$

$y_{n+2} = y_{n+1} + 4(y_n - y_{n+1}) - 6y_n$

$= -3y_{n+1} - 2y_n$

$\rightsquigarrow y^2 = -3y - 2 \longrightarrow y_1 = -1$, $y_2 = -2$

$y_n = d_1 \cdot (-1)^n + d_2 (-2)^n$

$y_1 = 1 \qquad y_2 = -5$

$$-d_1 - 2d_2 = 1$$
$$d_1 + 4d_2 = -5$$

$$2d_2 = -4$$
$$d_2 = -2$$
$$d_1 = 3$$

$$y_n = 3(-1)^n + (-2)^{n+1}$$

$$\Rightarrow a_n = \frac{x_n}{y_n} = \frac{(-1)^n - (-2)^n}{3\cdot(-1)^n + (-2)^{n+1}}.$$

⑥ Фибоначијев низ : $a_{n+2} = a_{n+1} + a_n$, $a_0 = a_1 = 1$

$$x^2 = x + 1 \longrightarrow x^2 - x - 1 = 0 \longrightarrow x_{1/2} = \frac{1 \pm \sqrt{5}}{2}$$

$x_1 \neq x_2$

$$a_n = c_1 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1+\sqrt{5}}{2}\right)^n$$

$c_1, c_2 = ?$ $\longrightarrow$ $a_0 = 1 = c_1 + c_2$

$$a_1 = 1 = c_1\left(\frac{1-\sqrt{5}}{2}\right) + c_2\left(\frac{1+\sqrt{5}}{2}\right) = (c_1 + c_2)\cdot\frac{1}{2} + \frac{\sqrt{5}}{2}(c_2 - c_1)$$

$$\sqrt{5}(c_2 - c_1) = 1$$

$$c_2 - c_1 = \frac{1}{\sqrt{5}} \qquad\qquad c_1 + c_2 = 1$$

$$> 2c_2 = 1 + \frac{1}{\sqrt{5}} \qquad c_2 = \frac{\sqrt{5}+5}{10} \qquad c_1 = \frac{5-\sqrt{5}}{10}$$

$$\frac{\sqrt{5}}{5}\left(\frac{\sqrt{5}-1}{2}\right)$$

$$''$$

$$\Rightarrow a_n = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{5+\sqrt{5}}{10}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$= -\frac{\sqrt{5}}{5}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

(БЧ) ✓

(ИК) → за вежбу.