- обичне диф. јне – ОДЈ, $x:\mathbb{R}\to\mathbb{R}$, $x(t)$
- парцијалне диф. јне – ПДЈ, $u:\mathbb{R}^n\to\mathbb{R}$, $u(x_1,\dots,x_n)$

Примери: $\Delta u=0$ – хармонијске фје

1) $u:\mathbb{R}^2\to\mathbb{R}$, $u(x,y)$ $\quad \Delta u=0 \iff \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

2. реда $\left(\dfrac{\partial^2}{\partial x^2}\text{ и }\dfrac{\partial^2}{\partial y^2}\right)$

2) $u,v:\mathbb{R}^2\to\mathbb{R}$

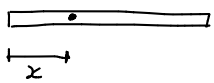
$$\left.\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}\right\}\text{ Коши–Риманове јне}$$

1. реда $\left(\dfrac{\partial}{\partial x}\text{ и }\dfrac{\partial}{\partial y}\right)$

3) Јна провођења топлоте: $u(t,x_1,\dots,x_n)$, $\quad u_t=k\Delta u$ (2. реда)

$$\Delta u=\frac{\partial^2 u}{\partial x_1^2}+\dots+\frac{\partial^2 u}{\partial x_n^2}$$

$u(x,t)$ – интензитет топлоте



t – време

Нотација:

$$\frac{\partial u}{\partial x}=u_x=u'_x$$

$$u_{xx}=u''_{xx}=\frac{\partial^2 u}{\partial x^2}$$

$$v_{xy}=v''_{xy}=\frac{\partial^2 v}{\partial x\partial y}$$

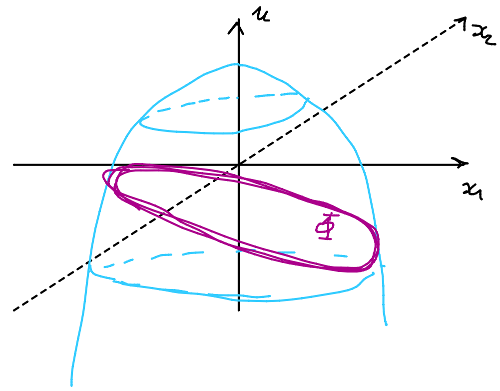
Општи облик ПДЈ 1. реда: $F\left(x_1,\dots,x_n,u,\dfrac{\partial u}{\partial x_1},\dots,\dfrac{\partial u}{\partial x_n}\right)=0$

$F:\mathbb{R}^{2n+1}\to\mathbb{R}$, $\quad u(x_1,\dots,x_n)=?$

Квазилинеарна (КЛ): $a_1(x_1,\dots,x_n,u)\dfrac{\partial u}{\partial x_1}+\dots+a_n(x_1,\dots,x_n,u)\dfrac{\partial u}{\partial x_n}=c(x_1,\dots,x_n,u)$

Кошијев проблем: наћи решење које садржи задату фигуру Φ

$\Phi\subseteq\Gamma(u)$
↳ график



Метода карактеристика

Посматрајмо КЛ: $\quad a(x,y,u)\dfrac{\partial u}{\partial x}+b(x,y,u)\dfrac{\partial u}{\partial y}=c(x,y,u)$ $\quad$ за $u(x,y)$

(ради једноставности $n=2$)

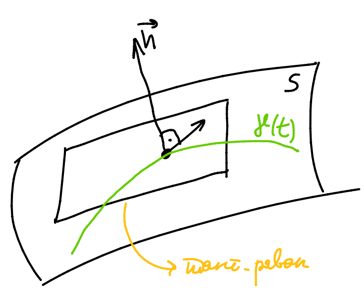
$S=\{(x,y,u(x,y))\}\subseteq\mathbb{R}^3$ → график решења $u(x,y)$ → површ у $\mathbb{R}^3$

$\vec{n}=(u_x(x,y),u_y(x,y),-1)$ – нормала на површ S у $(x,y,u(x,y))$

u решење $\Rightarrow a(x,y,u)\dfrac{\partial u}{\partial x}+b(x,y,u)\dfrac{\partial u}{\partial y}=c(x,y,u)$

$\Rightarrow a(x,y,u)\dfrac{\partial u}{\partial x}+b(x,y,u)\dfrac{\partial u}{\partial y}-c(x,y,u)=0$

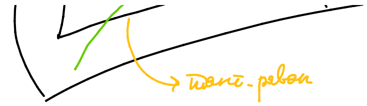
$\Rightarrow \langle(a,b,c),\underbrace{(u_x,u_y,-1)}_{\vec{n}}\rangle=0$



$$\Rightarrow \langle (a,b,c), \underbrace{(u_x, u_y, -1)}_{\vec{n}} \rangle = 0$$

$$\Rightarrow \langle (a,b,c), \vec{n} \rangle = 0$$

$\Rightarrow$ $(a,b,c) \in$ тангентној равни на S

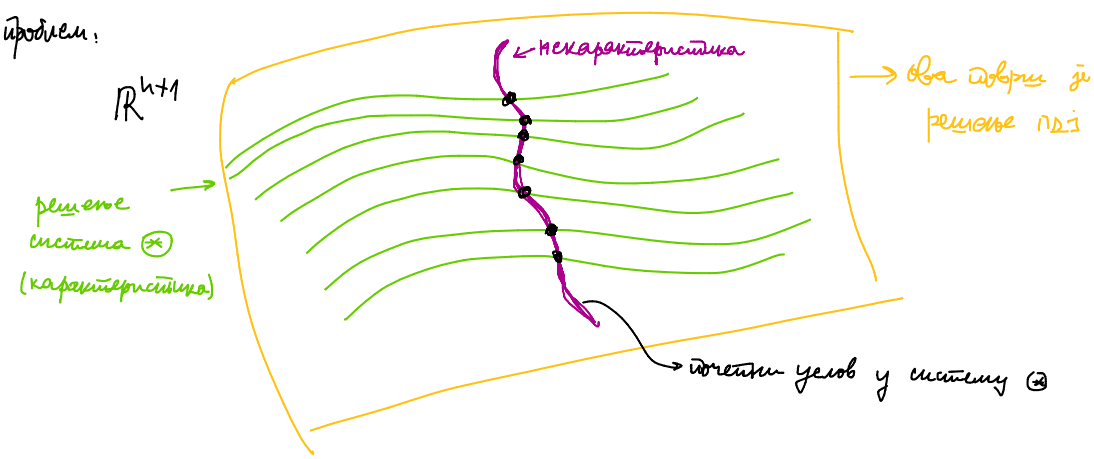


$$\Rightarrow \left.\begin{aligned} x'(t) &= a(x,y,u) \\ y'(t) &= b(x,y,u) \\ u'(t) &= c(x,y,u) \end{aligned}\right\} \otimes \text{ — систем карактеристика}$$

$\gamma(t) = (x(t), y(t), u(t))$ — решење система $\otimes$ $\Rightarrow$ $\gamma(t) \in S$, $\forall t$

$$\sum_{j=0}^{n} a_j(x_1, \dots, x_n, u) \frac{\partial u}{\partial x_j} = c(x_1, \dots, x_n, u) \Rightarrow \left.\begin{aligned} x_j'(t) &= a_j(x_1, \dots, x_n, u), \ \forall j \\ u'(t) &= c(x_1, \dots, x_n, u) \end{aligned}\right\}$$

Кошијев проблем:



(1) $u_x' + u_y' + 2u = 1 + u^2$, $u(\sin x, x + x^2) = x$

↳ Кошијев услов

$u(x,y)$

$$\underbrace{1}_{a_1} \cdot u_x' + \underbrace{1}_{a_2} \cdot u_y' = \underbrace{1 + u^2 - 2u}_{c} \Rightarrow \left.\begin{aligned} x'(t) &= 1 \\ y'(t) &= 1 \\ u'(t) &= 1 + u^2 - 2u \end{aligned}\right\}$$

$x' = 1 \Rightarrow x(t) = t + c_1$

$y' = 1 \Rightarrow y(t) = t + c_2$

$u' = (u-1)^2 \Rightarrow \frac{u'}{(u-1)^2} = 1$ (РП) $/ \int$

$$u' = (u-1)^2 \Rightarrow \frac{u'}{(u-1)^2} = 1 \quad {}^{(РП)} \Big/ \int$$

$$\int \frac{du}{(u-1)^2} = t + c_3$$

$$-\frac{1}{u-1} = t + c_3 \Rightarrow 1-u = \frac{1}{t+c_3} \Rightarrow u = 1 - \frac{1}{t+c_3} = \frac{t+c_3-1}{t+c_3}$$

$$u(\underbrace{\sin x}_{x}, \underbrace{x+x^2}_{y}) = \underbrace{x}_{u}$$ → Крива у $\mathbb{R}^3$ (xyu)

$$\gamma(s) = (\sin s, s+s^2, s) \subseteq \Gamma(u) \subseteq \mathbb{R}^3$$ → некарактеристика

$$c_1, c_2, c_3 \rightsquigarrow c_1(s), c_2(s), c_3(s)$$

$$x(t,s) = t + c_1(s)$$
$$y(t,s) = t + c_2(s)$$
$$u(t,s) = \frac{t + c_3(s) - 1}{t + c_3(s)}$$

Почетни услов:

$$x(0,s) = x_0(s) = \sin s$$
$$y(0,s) = y_0(s) = s + s^2$$
$$u(0,s) = u_0(s) = s$$

$$c_1(s) = \sin s$$
$$c_2(s) = s + s^2$$
$$\frac{c_3(s)-1}{c_3(s)} = s \Rightarrow c_3(s) = \frac{1}{1-s}$$

Површ (параметарски задата — s и t):

$$(x(t,s), y(t,s), u(t,s)) = \left(t + \sin s,\ t + s + s^2,\ \frac{t + \frac{1}{1-s} - 1}{t + \frac{1}{1-s}}\right) \checkmark \subseteq \mathbb{R}^3$$

② $(y+u)u'_x + y u'_y = x - y$, $\quad u|_{y=1} = x+1$

$u(x,y)$

$u(x,1) = x+1$, $\quad \gamma(s) = (s, 1, s+1) \subseteq \Gamma(u) \subseteq \mathbb{R}^3$

$$x' = y + u$$
$$y' = y$$
$$u' = x - y$$

$$x_0(s) = s$$
$$y_0(s) = 1$$
$$u_0(s) = s + 1$$

$$X = \begin{bmatrix} x \\ y \\ u \end{bmatrix}, \quad X' = AX$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$e^{tA} = P e^{tD} P^{-1}$$

ОР: $X(t) = e^{tA} \cdot c = P \cdot e^{tD} \cdot c_1 = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-t} & & \\ & e^t & \\ & & e^t \end{bmatrix} \cdot c_1 = \begin{bmatrix} -e^{-t} & e^t & e^t \\ 0 & e^t & 0 \\ e^{-t} & 0 & e^t \end{bmatrix} \cdot c_1$

$$X(t,s) = [\dots] \cdot c_1(s)$$

$$X(0,s) = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot c_1(s) \qquad = \qquad X(0,s) = \begin{bmatrix} x_0(s) \\ y_0(s) \\ u_0(s) \end{bmatrix} = \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix}$$

$$\Rightarrow c_1(s) = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix}$$

$$= \frac{1}{2} \cdot \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix}$$

$$X(t,s) = \begin{bmatrix} -e^{-t} & e^t & e^t \\ 0 & e^t & 0 \\ e^{-t} & 0 & e^t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} \overset{= x}{e^t(1+s) - e^{-t}} \\ \overset{= y}{e^t} \\ \overset{= u}{e^{-t} + se^t} \end{bmatrix}$$

решење: $(x, y, u) = (e^t(1+s) - e^{-t},\ e^t,\ e^{-t} + se^t)$ - површ у $\mathbb{R}^3$

следи: $e^t = y \Rightarrow t = \ln y$

$$e^{-t} = e^{-\ln y} = \frac{1}{y}$$

$$x = e^t(1+s) - e^{-t} = y(1+s) - \frac{1}{y} \Rightarrow s = \frac{x}{y} + \frac{1}{y^2} - 1$$

$$u(x,y) = e^{-t} + se^t = \frac{1}{y} + \left(\frac{x}{y} + \frac{1}{y^2} - 1\right) \cdot y = \underline{\underline{x - y + \frac{2}{y}}}.$$

провера: $(y+u)u'_x + yu'_y = x - y \quad , \quad u|_{y=1} = x+1$

$u|_{y=1} = x+1$

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = -1 - \frac{2}{y^2}$$

$u(x,1) = x - 1 + \frac{2}{1} = x+1$ ✓

$$\left(\cancel{y} + x - \cancel{y} + \frac{2}{y}\right)\cdot 1 + y\cdot\left(-1 - \frac{2}{y^2}\right) = x - y$$

$$x + \frac{2}{y} - y - \frac{2}{y} = x - y \ ✓$$

3. $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$ $\qquad (s,s,s^4) \subseteq \Gamma(u)$

(ХЛ), али посматрано као (КЛ)

$$\left.\begin{aligned} a_1(x,y,u) &= y \\ a_2(x,y,u) &= -x \\ c(x,y,u) &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x' &= y \\ y' &= -x \\ u' &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= c_1\cos t + c_2\sin t \\ y &= c_2\cos t - c_1\sin t \\ u &= c_3 \end{aligned}$$

$x_0(s) = s$

$y_0(s) = s$

$u_0(s) = s^4$

$$\Rightarrow \begin{aligned} c_1 &= s \\ c_2 &= s \\ c_3 &= s^4 \end{aligned}$$

површ: $\left(s(\cos t + \sin t),\ s(\cos t - \sin t),\ s^4\right)$

Γ стец: $x^2 + y^2 = s^2(\underbrace{\cos^2 t + \sin^2 t}_{1} + 2\cos t\sin t + \underbrace{\cos^2 t + \sin^2 t}_{1} - 2\cos t \sin t) = 2s^2$

$$\frac{(x^2+y^2)^2}{4} = s^4 = u$$