① $Y' = AY$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$$

$$k = 4$$

$$(A - 2E)\gamma = 0 \Rightarrow \ldots \; \gamma \in \underbrace{Lin\left\{ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\gamma_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\gamma_2} \right\}}_{\dim = 2} \Rightarrow m = 2 \Rightarrow 2 \text{ Ж. блока}$$

$$\varphi(\lambda) = \det(A - \lambda E) = (\lambda - 2)^4$$

$$\mu(\lambda) = (\lambda - 2)^k, \quad 1 \le k \le 4$$

$\mu(A) = 0$:

$$A - 2E \neq 0$$

$$(A - 2E)^2 \neq 0$$

$$(A - 2E)^3 = 0 \Rightarrow \deg \mu = 3 \Rightarrow \text{највећи ред блока је } 3 \Rightarrow \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & \\ & \boxed{2} \end{bmatrix} = J$$

$\gamma_1, \gamma_2$ - сопствени $\rightsquigarrow$ још 2 уопштена

уопштени за $\gamma_2$:

$$(A - 2E)\gamma = \gamma_2$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

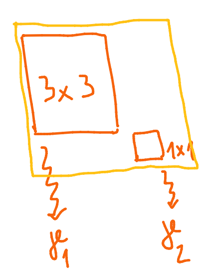
$$0 = 0$$

$$0 = 1 \;\; ↯ \;\; \Rightarrow \text{систем нема решења} \Rightarrow \gamma_2 \text{ нема уопштени}$$

$$d = 0 \qquad \Rightarrow \gamma_2 \text{ одговара блок } 1 \times 1$$

$$a = 0$$

уопштени за $\gamma_1$:




$$(A - 2E)\gamma = \gamma_1$$

$$\cancel{0 = 0}$$

$$\cancel{0 = 0} \Rightarrow \gamma = \begin{bmatrix} 0 \\ b \\ \ldots \end{bmatrix} \xrightarrow{b = c = 0} \gamma_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ldots \end{bmatrix}$$

$\cancel{0=0}$

$\cancel{0=0}$

$d=1$

$a=0$

$$\Rightarrow \gamma=\begin{bmatrix}0\\b\\c\\1\end{bmatrix} \xrightarrow{b=c=0} \gamma_3=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$\gamma_1 \rightsquigarrow \gamma_3$ (уточнени за $\gamma_1$) $\rightsquigarrow \gamma_4$ (уточнени за $\gamma_3$)

$\gamma_2$

уточнени за $\gamma_3$:

$(A-2E)\gamma=\gamma_3$

$\cancel{0=0}$

$\cancel{0=0}$

$d=0$

$a=1$

$$\Rightarrow \gamma=\begin{bmatrix}1\\b\\c\\0\end{bmatrix} \xrightarrow{b=c=0} \gamma_4=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$J=\begin{bmatrix}\boxed{\begin{matrix}2&1&\\&2&1\\&&2\end{matrix}}&\\&\boxed{2}\end{bmatrix}$$

$$T=\begin{bmatrix}\gamma_1&\gamma_3&\gamma_4&\gamma_2\end{bmatrix}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$$

$e^{xJ}=?$

$$J=\begin{bmatrix}B&\\&2\end{bmatrix}\Rightarrow e^{xJ}=\begin{bmatrix}e^{xB}&\\&e^{2x}\end{bmatrix}=e^{2x}\cdot\begin{bmatrix}1&x&\frac{x^2}{2}&\\&1&x&\\&&1&\\&&&1\end{bmatrix}$$

$$B=\begin{bmatrix}2&1&\\&2&1\\&&2\end{bmatrix}=\underbrace{\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}}_{D}+\underbrace{\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}}_{N}$$

$$N^2=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix},\ N^3=0$$

$$DN=ND\overset{(2)}{\Rightarrow} e^{xB}=e^{xD}\cdot e^{xN}=e^{2x}\cdot E\cdot\left(E+xN+\frac{x^2N^2}{2}\right)=e^{2x}\cdot\begin{bmatrix}1&x&\frac{x^2}{2}\\0&1&x\\0&0&1\end{bmatrix}$$

$OP: y(x)=Te^{xJ}\cdot c,\ c\in\mathbb{R}^4$

(2) $y'=Ay$

$$A=\begin{bmatrix}2&1&1\\0&1&-1\\0&1&3\end{bmatrix},\quad \varphi(\lambda)=(\lambda-2)^3$$

$\lambda_1=\lambda_2=\lambda_3=2,\ k=3$

$\dots\ x=a\cdot\begin{bmatrix}1\\0\end{bmatrix}+b\cdot\begin{bmatrix}0\\1\end{bmatrix}\Rightarrow m=\dim\ker(A-2E)=2\Rightarrow 2$ блока

$[0 \quad 1 \quad 3]$

$\lambda_1=\lambda_2=\lambda_3=2$, $k=3$

$$(A-2E)\gamma=0 \rightsquigarrow \gamma = a\cdot\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\gamma_1} + b\cdot\underbrace{\begin{bmatrix}0\\1\\-1\end{bmatrix}}_{\gamma_2} \Rightarrow m=\dim\ker(A-2E)=2 \Rightarrow 2 \text{ блока}$$

једина могућност: $J=\begin{bmatrix}2&1&\\&2&\\&&2\end{bmatrix}$

$$e^{xJ}=\dots=e^{2x}\cdot\begin{bmatrix}1&x&0\\0&1&0\\0&0&1\end{bmatrix}$$

$T=?$ имамо 2 сопс. век. $\rightsquigarrow$ 1 уопштени – да ли одговара $\gamma_1$ или $\gamma_2$?

$(A-2E)\gamma_3=\gamma_1$

$$\begin{bmatrix}0&1&1\\0&-1&-1\\0&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

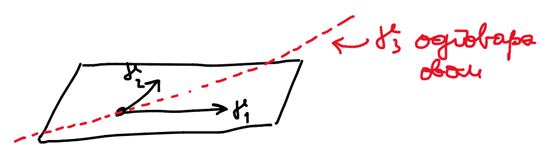
$b+c=1$
$-b-c=0$
$b+c=0$ $\lightning$

$(A-2E)\gamma_3=\gamma_2$

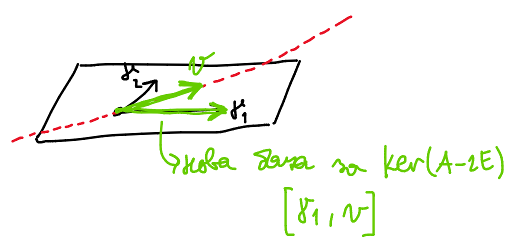
$$\begin{bmatrix}0&1&1\\0&-1&-1\\0&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$b+c=0$
$-b-c=1$
$b+c=-1$ $\lightning$

Сопствени потпростор:




идеја: променимо базу!




$v=?$, $v\in \mathcal{L}in\{\gamma_1,\gamma_2\}$, $v=\alpha\gamma_1+\beta\gamma_2$, $\alpha,\beta=?$

$v$ – има уопштени $\gamma_3$

$\exists$ решење једначине: $(A-2E)\gamma_3=v$

$$\begin{bmatrix}0&1&1\\0&-1&-1\\0&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\alpha\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}+\beta\cdot\begin{bmatrix}0\\1\\-1\end{bmatrix}=\begin{bmatrix}\alpha\\\beta\\-\beta\end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta\cdot\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ -\beta \end{bmatrix}$$

$$\left.\begin{aligned} b+c &= \alpha \\ -b-c &= \beta \\ \cancel{b+c} &\cancel{= -\beta} \end{aligned}\right\} \alpha = -\beta = 1$$

нпр. $b=1, c=0, a=0 \rightsquigarrow \gamma_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$v = \gamma_1 - \gamma_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{bmatrix} \rightsquigarrow T = \begin{bmatrix} v & \gamma_3 & \gamma_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow \text{OP}: y(x) = Te^{xJ}\cdot c,\ c\in\mathbb{R}^3$$

③ $y' = Ay$

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = 2, \quad \lambda_{3/4} = 1 \pm i$$

$\underline{\lambda_1 = \lambda_2 = 2:}$

$K=2$

$(A-2E)\gamma_1 = 0 \rightsquigarrow \gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow m=1 \Rightarrow$ 1 Жорд. блок и 1 усложнени

$\rightsquigarrow$ блок $\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$ у $J$

$(A-2E)\gamma_2 = \gamma_1$

$\vdots$

$\gamma_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda_{3/4} = 1 \pm i:}$

$1+i = \alpha + i\beta \Rightarrow \alpha = \beta = 1$

$\rightsquigarrow$ блок $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ у $J$

$(A-(1+i)E)\gamma_3 = 0$

$\vdots$

$$\gamma_3 = \begin{bmatrix} 1 \\ -2 \\ -1+i \\ -2i \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix} + i\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

$$J = \begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 1 & 1 \\ & & -1 & 1 \end{bmatrix} \rightsquigarrow e^{xJ} = \begin{bmatrix} e^{x\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}} & \\ & e^{x\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} e^{2x} & e^{2x}x & & \\ 0 & e^{2x} & & \\ & & e^{x}\cos x & e^{x}\sin x \\ & & -e^{x}\sin x & e^{x}\cos x \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} & & -1 & 1 \end{array}\right] \qquad \left[\; e^{x}\begin{bmatrix} & \\ -1 & 1\end{bmatrix}\right] \qquad \left[\; -e^{x}\sin x \quad e^{x}\cos x\right]$$

$\hookrightarrow e^{x}\cdot R_x$

$$T=\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$$\text{OP}:\; y(x)=Te^{xJ}c,\; c\in\mathbb{R}^4$$

④ $y'=Ay$

$$A=\begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\lambda_{1/2}=\lambda_{3/4}=\pm i$$

$$\lambda_1=+i,\; k=2 \quad \alpha+i\beta=i \rightsquigarrow \alpha=0,\beta=1 \rightsquigarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Могућности за $J$:

$$J=\begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{bmatrix} \quad \text{или} \quad J=\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}$$

$\hookrightarrow$ 2 Жорданова блока

$\hookrightarrow$ 1 Жорданов блок

$$(A-\lambda_1 E)\gamma_1^{c}=0$$

$$\gamma_1^{c}=\begin{bmatrix} 0 \\ -i \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}+i\cdot\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow m=\underbrace{\dim_{\mathbb{C}}}_{\text{комплексна димензија}} \operatorname{Ker}(A-\lambda_1 E)=1 \Rightarrow 1 \text{ Жорд. блок}$$

$$\Rightarrow J=\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Напомена: Ако би се добила 2 сопствена вектора: $\gamma_1^{c}, \gamma_2^{c}$

$$\Rightarrow m=2 \Rightarrow 2 \text{ Ж. блока} \Rightarrow J=\begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix} \rightsquigarrow e^{xJ}=\begin{bmatrix} e^{x\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} & \\ & e^{x\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \end{bmatrix}=\begin{bmatrix} R_x & \\ & R_x \end{bmatrix}$$

$$T=\left[\operatorname{Re}\gamma_1^{c}\downarrow \;\; \operatorname{Im}\gamma_1^{c}\downarrow \;\; \operatorname{Re}\gamma_2^{c}\downarrow \;\; \operatorname{Im}\gamma_2^{c}\downarrow\right]$$

$$T=\begin{bmatrix} \operatorname{Re}\gamma_1\downarrow & \operatorname{Im}\gamma_1\downarrow & \operatorname{Re}\gamma_2\downarrow & \operatorname{Im}\gamma_2\downarrow \end{bmatrix}$$

$\gamma_2$ – уточнители за $\gamma_1$

$$(A-\lambda_1 E)\gamma_2=\gamma_1$$

$$\Rightarrow \gamma_2=\begin{bmatrix}1\\0\\-1-i\\0\end{bmatrix}=\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}+i\cdot\begin{bmatrix}0\\0\\-1\\0\end{bmatrix}$$

$$T=\begin{bmatrix} \operatorname{Re}\gamma_1\downarrow & \operatorname{Im}\gamma_1\downarrow & \operatorname{Re}\gamma_2\downarrow & \operatorname{Im}\gamma_2\downarrow \end{bmatrix}=\begin{bmatrix}0&0&1&0\\0&-1&0&0\\0&0&-1&-1\\1&0&0&0\end{bmatrix}$$

$$J=\begin{bmatrix}A&E\\O&A\end{bmatrix}\quad\left(e^{xJ}\neq\begin{bmatrix}e^{xA}&e^{xE}\\e^{xO}&e^{xA}\end{bmatrix}\right)$$

$$A=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\quad E=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad O=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$J=\underset{D}{\begin{bmatrix}A&\\&A\end{bmatrix}}+\underset{N}{\begin{bmatrix}&E\\&\end{bmatrix}}$$

$$\left.\begin{aligned}DN&=\begin{bmatrix}A&\\&A\end{bmatrix}\begin{bmatrix}&E\\&\end{bmatrix}=\begin{bmatrix}O&A\\O&O\end{bmatrix}\\ ND&=\begin{bmatrix}&E\\&\end{bmatrix}\begin{bmatrix}A&\\&A\end{bmatrix}=\begin{bmatrix}O&A\\O&O\end{bmatrix}\end{aligned}\right\}\overset{2}{\Rightarrow} e^{xJ}=e^{xD}\cdot e^{xN}$$

$$\left.\begin{aligned}e^{xD}&=\begin{bmatrix}e^{xA}&\\&e^{xA}\end{bmatrix}=\begin{bmatrix}R_x&\\&R_x\end{bmatrix}\\ e^{xN}&=E+xN=\begin{bmatrix}1&0&x&0\\0&1&0&x\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}E&xE\\O&E\end{bmatrix}\end{aligned}\right\} e^{xJ}=\begin{bmatrix}R_x&\\&R_x\end{bmatrix}\cdot\begin{bmatrix}E&xE\\&E\end{bmatrix}=\begin{bmatrix}R_x&xR_x\\O&R_x\end{bmatrix}$$

$$N^2=\begin{bmatrix}O&E\\O&O\end{bmatrix}\begin{bmatrix}O&E\\O&O\end{bmatrix}=\begin{bmatrix}O&O\\O&O\end{bmatrix}\Rightarrow N^k=O,\ k\geq 2$$

$$\text{ОР}: y(x)=\begin{bmatrix}0&0&1&0\\0&-1&0&0\\0&0&-1&-1\\1&0&0&0\end{bmatrix}\begin{bmatrix}\cos x&\sin x&x\cos x&x\sin x\\-\sin x&\cos x&-x\sin x&x\cos x\\0&0&\cos x&\sin x\\0&0&-\sin x&\cos x\end{bmatrix}\cdot C,\quad C\in\mathbb{R}^4$$