## Нехомогени системи

$$\left(y' = p(x)y + g(x)\right)$$

$$Y'(x) = \underbrace{A\cdot Y(x)}_{\text{хомогени део}} + \underbrace{g(x)}_{\text{нехомогени део}}$$

$$\text{ОР}: \ Y(x) = \underbrace{Y_H(x)}_{\text{ОР хомогене}} + \underbrace{Y_p(x)}_{\text{ПР нехомогене}}$$

(*) $g(x) = P_s[x]\cdot e^{\mu x}$, $\mu \in \mathbb{R}$

$s \in \mathbb{N}_0$, $s = st\,(P_s[x]) = \deg(P_s[x])$

$\xrightarrow[\text{у облику}]{\text{тражимо}}$ $Y_p(x) = Q_{m+s}[x]\cdot e^{\mu x}$

$m$ = вишеструкост броја $\mu$ као решења карактеристичне јне. $A$

① a)

$$y_1' = 2y_1 + y_2 + xe^x$$
$$y_2' = -y_1 + 2y_2 - e^x$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad Y' = AY + g$$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} xe^x \\ -e^x \end{bmatrix} = \begin{bmatrix} x \\ -1 \end{bmatrix}\cdot e^x$$

ХОМОГЕНА: $Y' = AY$

$\det(A - \lambda E) = 0 \Rightarrow \lambda_{1/2} = 2 \pm i$

$\lambda_1 = 2 + i \rightsquigarrow (A - \lambda_1 E)\gamma_1 = 0 \rightsquigarrow \gamma_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\Psi(x) = e^{\lambda_1 x}\cdot \gamma_1 = e^{2x}\cdot \begin{bmatrix} \cos x + i\sin x \\ i\cos x - \sin x \end{bmatrix}$ $\rightsquigarrow \text{Re}$, $\rightsquigarrow \text{Im}$

ОРХ: $Y_H(x) = c_1 e^{2x}\cdot \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix} + c_2 e^{2x}\cdot \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}$, $c_1, c_2 \in \mathbb{R}$

НЕХОМОГЕНА:

$$g(x) = \begin{bmatrix} x \\ -1 \end{bmatrix}\cdot e^x = P_s[x]\cdot e^{\mu x}$$

$\mu = 1$, $s = 1$ $\rightsquigarrow$ $\mu$ није сопс.бр.$A$ ($\lambda_{1,2} = 2 \pm i$) $\Rightarrow m = 0$

$$\mu = 1 \quad \rightsquigarrow \quad \mu \text{ није сопс.вр. } A$$
$$s = 1 \qquad \lambda_{1/2} = 2 \pm i \qquad \Rightarrow m = 0$$

$$y_p(x) = Q_{0+1}[x] \cdot e^{1x} = Q_1[x] \cdot e^x = \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} e^x \rightsquigarrow y_p' = A y_p + g$$

$$\begin{bmatrix} a_1 x + b_1 + a_1 \\ a_2 x + b_2 + a_2 \end{bmatrix} \cdot \cancel{e^x} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \cdot \cancel{e^x} + \begin{bmatrix} x \\ -1 \end{bmatrix} \cdot \cancel{e^x}$$

$$\left.\begin{aligned} a_1 x + b_1 + a_1 &= 2a_1 x + 2b_1 + a_2 x + b_2 + x \\ a_2 x + b_2 + a_2 &= -a_1 x - b_1 + 2a_2 x + 2b_2 - 1 \end{aligned}\right\} \forall x \Rightarrow$$

$$\begin{aligned} a_1 &= 2a_1 + a_2 + 1 \\ b_1 + a_1 &= 2b_1 + b_2 \\ a_2 &= -a_1 + 2a_2 \rightsquigarrow a_2 = a_1 \\ b_2 + a_2 &= -b_1 + 2b_2 - 1 \end{aligned} \qquad \Rightarrow a_1 = a_2 = -\frac{1}{2}$$

$$\left.\begin{aligned} -\frac{1}{2} &= b_1 + b_2 \\ \frac{1}{2} &= -b_1 + b_2 \end{aligned}\right\} b_2 = 0, \ b_1 = -\frac{1}{2}$$

$$y_p(x) = \begin{bmatrix} -\frac{1}{2}x - \frac{1}{2} \\ -\frac{1}{2}x \end{bmatrix} \cdot e^x$$

ОР: $y(x) = y_H(x) + y_p(x)$

б) $y_1' = 2y_1 + y_2 + 2e^x$

$y_2' = y_1 + 2y_2 - 3e^{4x}$

$$y' = Ay + g$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 2e^x \\ -3e^{4x} \end{bmatrix}$$

ХОМ: $\lambda_1 = 1, \ \lambda_2 = 3$

$$\gamma_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \gamma_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\varphi_1(x) = e^x \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \varphi_2(x) = e^{3x} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

НЕХОМ: $g(x) = \begin{bmatrix} 2e^x \\ -3e^{4x} \end{bmatrix} \neq P_s[x] \cdot e^{\mu x}$

$$g(x) = g_1(x) + g_2(x)$$

$$\left.\begin{array}{l} g_1(x) = \begin{bmatrix} 2e^{x} \\ 0 \end{bmatrix} ; \quad g_2(x) = \begin{bmatrix} 0 \\ -3e^{4x} \end{bmatrix} \\ \quad \rightsquigarrow y_{P_1}(x) \qquad\qquad \rightsquigarrow y_{P_2}(x) \end{array}\right\} y_P(x) = y_{P_1}(x) + y_{P_2}(x)$$

$\underline{g_1(x):}$

$$g_1(x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot e^{x} = P_s[x] \cdot e^{\mu x}$$

$$\begin{array}{l} \mu = 1 \\ s = 0 \end{array} \xrightarrow[\mu=1]{} m = 1 \Rightarrow y_{P_1}(x) = \underset{s+m=1}{Q_1}[x] \cdot e^{x} = \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \cdot e^{x}$$

$$y_{P_1}' = A y_{P_1} + g_1(x)$$

$$y_{P_1}(x) \underset{\text{нпр.}}{=} \begin{bmatrix} x-1 \\ -x \end{bmatrix} \cdot e^{x}$$

$\underline{g_2(x):}$

$$g_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot e^{4x} \Rightarrow \begin{array}{l} s = 0 \\ \mu = 4 \rightsquigarrow m = 0 \end{array} \Rightarrow y_{P_2}(x) = Q_0[x] \cdot e^{4x} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot e^{4x}$$

$$y_{P_2}' = A y_{P_2} + g_2(x)$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot 4\cancel{e^{4x}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \cdot \cancel{e^{4x}} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot \cancel{e^{4x}}$$

$$\left.\begin{array}{l} 4a = 2a + b \\ 4b = a + 2b - 3 \end{array}\right\} \begin{array}{l} a = -1 \\ b = -2 \end{array}$$

$$y_{P_2}(x) = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot e^{4x}$$

$$\text{ОР}, \ y(x) = c_1 \cdot \begin{bmatrix} e^{x} \\ -e^{x} \end{bmatrix} + c_2 \cdot \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix} + \begin{bmatrix} x-1 \\ -x \end{bmatrix} \cdot e^{x} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot e^{4x}, \quad c_1, c_2 \in \mathbb{R}$$

$y' = A(x) \cdot y$ не мора обавезно да буде са КК

ОР: $y(x) = \underline{\Phi(x) \cdot c}, \quad c \in \mathbb{R}^n$

$$\Phi(x) = \begin{bmatrix} \varphi_1 \downarrow & \cdots & \varphi_n \downarrow \end{bmatrix}$$

$$y' = A(x) \cdot y + g(x)$$

$\Phi(x)$ – фунд. матр. хомогеног система $y' = A(x) \cdot y$

ОР: $y(x) = \Phi(x) \cdot \left( c + \int \Phi^{-1}(x) \cdot g(x)\, dx \right) = \underbrace{\Phi(x) \cdot c} + \underbrace{\Phi(x) \cdot \int \Phi^{-1}(x) \cdot g(x)\, dx}$

ОР: $y(x) = \Phi(x)\cdot\left(C + \int \Phi^{-1}(x)\cdot g(x)\,dx\right) = \underbrace{\Phi(x)\cdot C}_{\text{ОРХ}} + \underbrace{\Phi(x)\cdot\int\Phi^{-1}(x)\cdot g(x)\,dx}_{y_p(x)\ \text{од нехомогене}}$

(2) $y_1' = y_2 + tg^2x + 1$

$y_2' = -y_1 + tg\,x$

$$y' = Ay + g(x)$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad g(x) = \begin{bmatrix} tg^2x + 1 \\ tg\,x \end{bmatrix} \ ?\ \cancel{=}\ P_s[x]\cdot e^{\mu x}$$

$$\Phi(x) = ?$$

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned}\right\}\quad y_1'' = y_2' = -y_1 \Rightarrow y_1'' + y_1 = 0 \rightsquigarrow y_1(x) = c_1\cos x + c_2\sin x$$

$$y_2(x) = -c_1\sin x + c_2\cos x$$

$$y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1\cdot\underbrace{\begin{bmatrix} \cos x \\ -\sin x \end{bmatrix}}_{\varphi_1(x)} + c_2\cdot\underbrace{\begin{bmatrix} \sin x \\ \cos x \end{bmatrix}}_{\varphi_2(x)}$$

$$\Phi(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

$\varphi_1(x)$ $\varphi_2(x)$

$$W(x) = \det\Phi(x) = \cos^2x - (-\sin^2x) = 1 \neq 0$$

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \rightsquigarrow A^{-1} = \frac{1}{\det A}\cdot\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \Rightarrow \Phi^{-1}(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$\int\Phi^{-1}(x)\cdot g(x)\,dx = \int\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}\cdot\begin{bmatrix} tg^2x + 1 \\ tg\,x \end{bmatrix}dx = \int\begin{bmatrix} \frac{\sin^2x}{\cos x} + \cos x - \frac{\sin^2x}{\cos x} \\ \frac{\sin^3x}{\cos^2x} + \sin x + \sin x \end{bmatrix}dx =$$

Члан по члан

$$= \int\begin{bmatrix} \cos x \\ \frac{\sin^3x}{\cos^2x} + 2\sin x \end{bmatrix}dx = \begin{bmatrix} \int\cos x\,dx \\ \int\left(\frac{\sin^3x}{\cos^2x} + 2\sin x\right)dx \end{bmatrix} = \begin{bmatrix} \sin x \\ \cos x + \frac{1}{\cos x} - 2\cos x \end{bmatrix} = \begin{bmatrix} \sin x \\ -\cos x + \frac{1}{\cos x} \end{bmatrix}$$

$$\int\frac{\sin^3x}{\cos^2x}dx = \int\frac{(1-\cos^2x)\cdot\sin x\,dx}{\cos^2x} = \int\frac{\cos^2x - 1}{\cos^2x}(-\sin x)\,dx = \int\frac{t^2-1}{t^2}dt = t - \int\frac{dt}{t^2} =$$

$\cos x = t$

$dt = (-\sin x)\,dx$

$$= t - \left(-\frac{1}{t}\right) = t + \frac{1}{t} = \cos x + \frac{1}{\cos x}$$

$$y_p(x) = \Phi(x)\cdot\int\Phi^{-1}(x)g(x)\,dx = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\cdot\begin{bmatrix} \sin x \\ -\cos x + \frac{1}{\cos x} \end{bmatrix} = \begin{bmatrix} \cancel{\sin x\cos x} - \cancel{\sin x\cos x} + \frac{\sin x}{\cos x} \\ \underbrace{-\sin^2x - \cos^2x}_{-1} + \cancel{1} \end{bmatrix} = \begin{bmatrix} tg\,x \\ 0 \end{bmatrix}$$

$$y_p(x) = \Phi(x)\cdot\int \Phi^{-1}(x) g(x)\,dx = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} \cdots \\ -\cos x + \frac{1}{\cos x} \end{bmatrix} = \begin{bmatrix} \cdots \\ \underbrace{-\sin^2 x - \cos^2 x}_{-1} + 1 \end{bmatrix} = \begin{bmatrix} \cdots \\ 0 \end{bmatrix}$$

$$\text{OP}: \ y(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \operatorname{tg} x \\ 0 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

## Експоненти матрице

$n \in \mathbb{N}$, $A \in M_n(\mathbb{R})$ $(\mathbb{C})$ $\rightsquigarrow \exp: M_n(\mathbb{R}) \to M_n(\mathbb{R})$

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$A \in M_n(\mathbb{R}) \Rightarrow A^k \in M_n(\mathbb{R})$

$$\left[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \ x \in \mathbb{R} \right]$$

**Особине:** $(\forall A, B \in M_n(\mathbb{R}))$

1. $\frac{d}{dt}(e^{tA}) = A \cdot e^{tA} = e^{tA} \cdot A$
2. $AB = BA \Rightarrow e^{A+B} = e^A \cdot e^B$
3. $AB = BA \Rightarrow Ae^B = e^B A$ (специј. $AA = AA = A^2 \Rightarrow Ae^A = e^A A$)
4. $\det(e^A) = e^{\operatorname{tr} A}$
5. $\lim_{n\to\infty} \left(E + \frac{A}{n}\right)^n = e^A$ $\left(1 + \frac{x}{n}\right)^n \xrightarrow[n\to\infty]{} e^x$

**3** Испитати да ли постоји $A \in M_2(\mathbb{R})$ тд:

$(e^a = b, \ b > 0$ – има решења$)$

а) $e^A = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ б) $e^A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$

а) $\det(e^A) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}\right) = -4$

④ $\Rightarrow \det(e^A) = e^{\operatorname{tr} A}$

$\Big\}\ e^{\operatorname{tr} A} = -4$ ↯ $\Rightarrow \nexists A$

б) $\det(e^A) = \det\left(\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\right) = 4 > 0$

$$e^{trA} = 4 \Rightarrow trA = \ln 4$$

$$AA = AA = A^2 \overset{(3)}{\Longrightarrow} Ae^A = e^A A$$

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \rightsquigarrow \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$\cancel{-\alpha = -\alpha}$$
$$\left.\begin{array}{l} -4\beta = -\beta \Rightarrow 3\beta = 0 \\ -\gamma = -4\gamma \Rightarrow 3\gamma = 0 \end{array}\right\} \beta = \gamma = 0$$
$$\cancel{-4\delta = -4\delta}$$

$$\Rightarrow A = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}, A^2 = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \delta^2 \end{bmatrix}, A^3 = \begin{bmatrix} \alpha^3 & 0 \\ 0 & \delta^3 \end{bmatrix}, \dots, A^k = \begin{bmatrix} \alpha^k & 0 \\ 0 & \delta^k \end{bmatrix}$$

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{bmatrix} \alpha^k & 0 \\ 0 & \delta^k \end{bmatrix}}{k!} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{\delta^k}{k!} \end{bmatrix} = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^\delta \end{bmatrix}$$

$$e^A = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^\delta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \Rightarrow \left.\begin{array}{l} e^\alpha = -1 \\ e^\delta = -4 \end{array}\right\} \lightning \Rightarrow \nexists A$$

(*) $A = \mathrm{diag}[\lambda_1\ \lambda_2 \dots \lambda_n] = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Rightarrow e^A = \mathrm{diag}[e^{\lambda_1} \dots e^{\lambda_n}]$

(4) Нека су $A, B \in M_n(\mathbb{R})$ и нека $\exists B^{-1}$. Доказати $e^{B^{-1}AB} = B^{-1}e^A B$.

$$e^{B^{-1}AB} = \sum_{k=0}^{\infty} \frac{(B^{-1}AB)^k}{k!} = \sum_{k=0}^{\infty} \frac{B^{-1} \cdot A^k \cdot B}{k!} = B^{-1} \cdot \left( \sum_{k=0}^{\infty} \frac{A^k}{k!} \right) \cdot B = B^{-1} \cdot e^A \cdot B.$$

$$(B^{-1}AB)^k = (B^{-1}AB) \cdot (B^{-1}AB) \cdot (B^{-1}AB) \cdot \dots \cdot (B^{-1}AB) =$$
$$= B^{-1}A\underbrace{(BB^{-1})}_{E} \cdot A \cdot \underbrace{(BB^{-1})}_{E} \cdot A \cdot (\ ) \dots \underbrace{(BB^{-1})}_{E} AB =$$
$$= B^{-1} \cdot A \cdot A \cdot A \cdot \dots \cdot AB =$$
$$= B^{-1}A^kB$$

(*) множење матрица није комутативно, али јесте асоцијативно