① Свести систем ДЈ на систем ДЈ у нормалном облику.

$y' = f(x,y)$

↳ колекција неколико ДЈ

$y_1, y_2, \dots, y_n$ → непознате ф-је

$x$ → независна променљива

→ систем јна у које фигурише

$x, y_j, y_j', y_j'', y_j''', \dots$

$$y_1' = f_1(x, y_1, \dots, y_n)$$
$$y_2' = f_2(x, y_1, \dots, y_n)$$
$$\vdots$$
$$y_n' = f_n(x, y_1, \dots, y_n)$$

$$\begin{cases} y''' = xy'' + y + z' + 2x - 1 \\ z'' = y' \cdot \sin x + \cos(y \cdot z') \end{cases}$$

→ 3. реда → 3 јна 1. реда

→ 2. реда → 2 јна 1. реда

5 јна 1. реда

за непознате: $y, z$ и још: $y', y''$, $z'$

даје 2 јне:

$$y_2' = x \cdot y_2 + y + z_1 + 2x - 1$$
$$z_1' = y_1 \cdot \sin x + \cos(y \cdot z_1)$$

Где су још 3 јне?

добијамо из ових услова

$y$

$y_1 = y'$

$y_2 = y''$

$z$

$z_1 = z'$

$$y' = y_1$$
$$y_1' = (y')' = y'' = y_2$$
$$z' = z_1$$

$$\left.\begin{aligned} y' &= y_1 \\ y_1' &= y_2 \\ y' &= xy_2 + y + z_1 + 2x - 1 \\ z' &= z_1 \\ z_1' &= y_1 \cdot \sin x + \cos(y \cdot z_1) \end{aligned}\right\} \text{систем реда } 5$$

② Методом елиминације решити системе ДЈ:

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а) $y' = py - qz$
$\quad z' = qy + pz$ $\qquad p, q \in \mathbb{R} \setminus \{0\}$

б) $y_1' = y_2$
$\quad y_2' = y_1$
$\quad y_3' = y_1 + y_2 + y_3$

изразити из једне јне
и убацити у другу

а) систем 2 дј → 1 дј реда 2

$$y' = py - qz \Rightarrow qz = py - y' \,/ : q \,(\neq 0)$$

$$z = \frac{py - y'}{q} \,/\,' \Rightarrow z' = \frac{py' - y''}{q}$$

$$z' = qy + pz$$

$$\frac{py' - y''}{q} = q \cdot y + p \cdot \frac{py - y'}{q} \,/ \cdot q$$

$$py' - y'' = q^2 y + p^2 y - py'$$

$$y'' - 2py' + (p^2 + q^2) y = 0 \rightarrow \text{ЛДЈ 2. реда}$$

$$\lambda^2 - 2p\lambda + (p^2 + q^2) = 0$$

$$D = (-2p)^2 - 4(p^2 + q^2) =$$
$$= -4q^2 < 0$$

$$\lambda_{1/2} = \frac{2p \pm i \cdot 2q}{2} = p \pm iq$$

$$\Rightarrow \text{ОР}: \quad y(x) = c_1 e^{px} \cos qx + c_2 e^{px} \sin qx, \quad c_1, c_2 \in \mathbb{R}$$

$$z = \frac{py - y'}{q} = \ldots = c_1 e^{px} \sin qx - c_2 e^{px} \cos qx$$

$$y' = c_1 (e^{px})' \cdot \cos qx + c_1 e^{px} \cdot (\cos qx)' + c_2 (e^{px})' \sin qx + c_2 e^{px} (\sin qx)' =$$
$$= c_1 p e^{px} \cos qx - c_1 q e^{px} \sin qx + c_2 p e^{px} \sin qx + c_2 q e^{px} \cos qx$$

б) $\left. \begin{array}{l} y_1' = y_2 \\ y_2' = y_1 \end{array} \right\} \rightarrow$ 2 јне, 2 непознате

$\quad y_3' = y_1 + y_2 + y_3$

$$y_2 = y_1' \Rightarrow y_2' = y_1''$$

$y_3' = y_1 + y_2 + y_3$

$$y_2 = y_1' \Rightarrow y_2' = y_1''$$

$$y_2' = y_1 \Rightarrow y_1'' = y_1 \overset{\lambda^2 - 1 = 0}{\Rightarrow} y_1(x) = c_1 e^{x} + c_2 e^{-x}, \quad \underline{\underline{c_1, c_2 \in \mathbb{R}}}$$

$$y_2(x) = y_1'(x) = c_1 e^{x} - c_2 e^{-x}$$

$$y_3' - y_3 = y_1 + y_2 = 2c_1 e^{x}$$

$p(x) = -1$

$g(x) = 2c_1 e^{x}$

$\int p(x)\,dx = -x$

$\int g(x) \cdot e^{\int p(x)dx} dx = \int 2c_1 e^{x} \cdot e^{-x} dx = \int 2c_1 dx = 2c_1 x$

$$y_3 = e^{-\int p(x) dx} \cdot \left( c_3 + \int g(x) \cdot e^{\int p(x) dx} dx \right) = e^{x} \cdot (c_3 + 2c_1 x)$$

$c_3 \in \mathbb{R}$

③ Помоћу одговарајуће смене независне променљиве решити систем ДЈ:

а) $\boxed{2\sqrt{x}} \cdot y' = 2y - z$

$\boxed{2\sqrt{x}} \cdot z' = y + 2z$

б) $\boxed{x} y_1' = y_1 + y_2$

$\boxed{x} y_2' = y_2$

$\boxed{x} y_3' = -y_3$

напомена б:

$x y_2' = y_2$

$\frac{y_2'}{y_2} = \frac{1}{x}$ (РП) ...

$\frac{y_3'}{y_3} = -\frac{1}{x}$ (РП) ...

могло је и директно

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \quad Y : \mathbb{R} \to \mathbb{R}^n$$

$$Y' = \underset{(f_1, \dots, f_n)}{\underset{\parallel}{F}}(x, Y), \quad F : \mathbb{R}^{n+1} \to \mathbb{R}^n$$

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$f(x) \cdot Y' = F(x, Y)$

$f(x) = 2\sqrt{x}, x$

$x \rightsquigarrow t \qquad \frac{d}{dx} Y \rightsquigarrow \frac{d}{dt} Y$

$x(t), t(x)$

Желимо: $f(x) \cdot \frac{d}{dx} Y = \frac{d}{dt} Y$

$y_j$ гледамо: $f(x) \cdot \frac{dy_j}{dx} = \frac{dy_j}{dt} = \frac{dy_j}{dx} \cdot \frac{dx}{dt}$

$f(x) = \frac{dx}{dt} \Rightarrow \frac{dt}{dx} = \frac{1}{f(x)}$ (РП)

$$f(x) = \frac{dx}{dt} \Rightarrow \frac{dt}{dx} = \frac{1}{f(x)} \quad \text{(pn)}$$

$$\boxed{t = \int \frac{dx}{f(x)}}$$

a) $\frac{dt}{dx} = \frac{1}{f(x)} = \frac{1}{2\sqrt{x}} \Rightarrow t = \int \frac{dx}{2\sqrt{x}} = \sqrt{x}$

$$t(x) = \sqrt{x}, \quad x(t) = t^2 \qquad \frac{dt}{dx} = \frac{1}{2\sqrt{x}}$$

$$2\sqrt{x} \cdot y' = 2\sqrt{x} \cdot \frac{dy}{dx} = 2\sqrt{x} \cdot \frac{dy}{dt} \cdot \frac{dt}{dx} = 2\sqrt{x} \cdot \frac{dy}{dt} \cdot \frac{1}{2\sqrt{x}} = \frac{dy}{dt}$$

$$y_t' = \frac{dy}{dt} = 2y - z \Rightarrow z = 2y - y_t' \Rightarrow z_t' = 2y_t' - y_t''$$

$$z_t' = \frac{dz}{dt} = y + 2z$$

$$\Rightarrow 2y_t' - y_t'' = y + 4y - 2y_t'$$

$$y_t'' - 4y_t' + 5y = 0$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$D = 16 - 4 \cdot 5 = -4 < 0$$

$$\lambda_{1/2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$y(t) = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t, \quad c_1, c_2 \in \mathbb{R}$$

$$y_t'(t) = \ldots$$

$$z(t) = 2y - y_t' = \ldots$$

OP: $\left.\begin{aligned} y(x) &= c_1 e^{2\sqrt{x}} \cos\sqrt{x} + c_2 e^{2\sqrt{x}} \sin\sqrt{x} \\ z(x) &= -c_2 e^{2\sqrt{x}} \cos\sqrt{x} + c_1 e^{2\sqrt{x}} \sin\sqrt{x} \end{aligned}\right\} \quad c_1, c_2 \in \mathbb{R}, \quad x > 0$

$x = 0$: $\left.\begin{aligned} 2y(0) - z(0) &= 0 \\ y(0) + 2z(0) &= 0 \end{aligned}\right\} \quad y(0) = z(0) = 0$ ⇒ не могу се сва решења додефинисати у нули б) $xy_1' = y_1+y_2$

$xy_2' = y_2$

$xy_3' = -y_3$

$x=0:\ y_1(0)+y_2(0) = y_2(0) = -y_3(0) = 0$

$\Rightarrow y_1(0)=y_2(0)=y_3(0)=0$

$$\frac{dt}{dx} = \frac{1}{x}$$

$$t = \int \frac{dx}{x} = \ln|x|$$

$1^\circ$ $x>0$, $t=\ln x$, $x=e^t$

$$\frac{dy_1}{dx}\cdot x = \frac{dy_1}{dt}\cdot \cancel{\frac{dt}{dx}}^{\,=\frac{1}{x}}\cdot \cancel{x} = \frac{dy_1}{dt}$$

$y_{1t}' = y_1+y_2$

$y_{2t}' = y_2 \Rightarrow y_2(t) = c_1 e^{t}$ $\quad e^t \to x$

$y_{3t}' = -y_3 \Rightarrow y_3(t) = c_2 e^{-t}$ $\quad e^{-t} \to \frac{1}{x}$

$y_{1t}' = y_1 + c_1 e^t$

$\vdots$

$y_1(t) = e^t(c_3+c_1 t)$ $\quad e^t \to x,\ t \to \ln x$

$$\text{ОР}:\ Y(x) = \begin{bmatrix} x(c_3+c_1\ln x) \\ c_1 x \\ \frac{c_2}{x} \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \in \mathbb{R}^3$$

$2^\circ$ $x<0$, $t=\ln(-x)$, $x=-e^t$

$\vdots$

домаћи

(4) Методом елиминације решити систем ДЈ:

а) $y' = 1-\frac{1}{z}$

$z' = \frac{1}{y-x}$

б) $2zy' = y^2-z^2+1$

$z' = y+z$

а) $z\neq 0$

$y \neq x$

$$\frac{1}{z'} = y-x \Rightarrow y = x+\frac{1}{z'} \ \Big/'$$

$z'\neq 0$

$$y' = 1+\frac{d}{dx}\left(\frac{1}{z'}\right) = 1+\left(-\frac{1}{z'^2}\right)\cdot z'' = 1-\frac{z''}{(z')^2}$$

$$y' = 1-\frac{1}{z}$$

$$\cancel{1}+\frac{z''}{(z')^2}=\cancel{1}+\frac{1}{z} \Rightarrow (z')^2=z''\cdot z \;/:z^2$$ ← нелинеарна 2. реда (нема $x$)

$$\frac{z''\cdot z-(z')^2}{z^2}=0$$

$$\left(\frac{z'}{z}\right)'=\frac{(z')'\cdot z-z'\cdot(z)'}{z^2}=$$
$$=\frac{z''\cdot z-(z')^2}{z^2}$$

$$\left(\frac{z'}{z}\right)'=0$$

$$\frac{z'}{z}=c_1 \;/\int \qquad , c_1\in\mathbb{R}$$

$$\ln|z|=c_1x+c_2$$

$$|z|=e^{c_1x}\cdot e^{c_2}$$

$$\boxed{z=c_3\cdot e^{c_1x}} \qquad , c_3\in\mathbb{R}\setminus\{0\}$$

$$z'=c_3\cdot c_1\cdot e^{c_1x}$$
$$c_1\neq 0$$

$$\boxed{y=x+\frac{1}{c_1c_3e^{c_1x}}}$$

б) $2zy'=y^2-z^2+1$

$$z'=y+z \longrightarrow y=z'-z$$
$$y'=z''-z'$$

$$2z\cdot(z''-z')=(z'-z)^2-z^2+1$$

$$2zz''-2zz'=z'^2-2zz'+z^2-z^2+1$$

$$2zz''=z'^2+1$$ ← нелин. 2. реда (нема $x$)

СМЕНА: $u(z)=z'$

$$u'(z)=\frac{d}{dz}(z')=\frac{dz'}{dx}\cdot\frac{dx}{dz}=z''\cdot\frac{1}{\frac{dz}{dx}}=\frac{z''}{z'}$$

$$z''=u'\cdot z'=u'u$$

$$2\cdot z\cdot u'u=u^2+1$$

$$\frac{2u'u}{u^2+1}=\frac{1}{z} \quad \text{(РП)}$$

$$\int\frac{2u\,du}{u^2+1}=\int\frac{dz}{z}$$

$$\ln(u^2+1)=\ln|z|+c_1 \quad , c_1\in\mathbb{R}$$

$$u^2+1=|z|\cdot e^{c_1}=c_2\cdot z \quad , c_2\in\mathbb{R}\setminus\{0\}$$

$$z'^2+1=c_2z$$

$$z'^2 = c_2 z - 1 \qquad (c_2 z - 1 \geq 0)$$

$$z' = \pm\sqrt{c_2 z - 1}$$

$1^\circ$ $z' = \sqrt{c_2 z - 1}$

$$\frac{dz}{\sqrt{c_2 z - 1}} = dx \;\Big/ \int$$

$$\frac{1}{c_2} 2\sqrt{c_2 z - 1} = x + c_3$$

$$\Downarrow$$

$$\vdots$$

$$z = \frac{1}{c_2}\left(1 + \frac{c_2^2}{4}(x+c_3)^2\right) \Rightarrow z' = \ldots = \frac{c_2}{2}(x+c_3)$$

$$y = z' - z = \ldots = \frac{c_2}{2}(x+c_3) - \frac{1}{c_2} - \frac{c_2}{4}(x+c_3)^2$$

$2^\circ$ $z' = -\sqrt{c_2 z - 1}$

$\vdots$

напомена: НЕ враћати $y(x)$ као $y' = \frac{y^2 - z^2 + 1}{2z} = \frac{(z'-z)^2 - z^2 + 1}{2z}$

или

$y' = z'' - z'$

јер ће то дати још 1 константу

систем 2. реда → зависи ОР од 2 константе