Парцијалне ДЈ 1. реда

- обичне диф. јне – ОДЈ , $x:\mathbb{R}\to\mathbb{R}$, $x(t)$
- парцијалне диф. јне – ПДЈ , $u:\mathbb{R}^n\to\mathbb{R}$, $u(x_1,\dots,x_n)$

Примери: $\Delta u = 0$ – хармонијске фје

1) $u:\mathbb{R}^2\to\mathbb{R}$, $u(x,y)$ $\qquad \Delta u = 0 \iff \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ $\quad$ 2. реда $\left(\frac{\partial^2}{\partial x^2} \text{ и } \frac{\partial^2}{\partial y^2}\right)$

2) $u,v:\mathbb{R}^2\to\mathbb{R}$

$$\left.\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}\right\} \text{ Коши–Риманове јне}$$

1. реда $\left(\frac{\partial}{\partial x} \text{ и } \frac{\partial}{\partial y}\right)$

3) Јна провођења топлоте: $u(t,x_1,\dots,x_n)$, $u_t = k\Delta u$ $\quad$ 2. реда

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2}+\dots+\frac{\partial^2 u}{\partial x_n^2}$$

нотација: $\frac{\partial u}{\partial x} = u_x = u'_x$

Општи облик ПДЈ 1. реда: $F\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right)=0$

$F:\mathbb{R}^{2n+1}\to\mathbb{R}$, $u(x_1,\dots,x_n)=?$

- Хомогена линеарна (ХЛ): $a_1(x_1,\dots,x_n)\frac{\partial u}{\partial x_1}+\dots+a_n(x_1,\dots,x_n)\frac{\partial u}{\partial x_n}=0$
- линеарна (Л): $a_1(x_1,\dots,x_n)\frac{\partial u}{\partial x_1}+\dots+a_n(x_1,\dots,x_n)\frac{\partial u}{\partial x_n}+a_{n+1}(x_1,\dots,x_n)u = c(x_1,\dots,x_n)$
- квазилинеарна (КЛ): $a_1(x_1,\dots,x_n,\underline{u})\frac{\partial u}{\partial x_1}+\dots+a_n(x_1,\dots,x_n,\underline{u})\frac{\partial u}{\partial x_n} = c(x_1,\dots,x_n,\underline{u})$

$$(\text{ХЛ})\subseteq(\text{Л})\subseteq(\text{КЛ})$$

Метода карактеристика

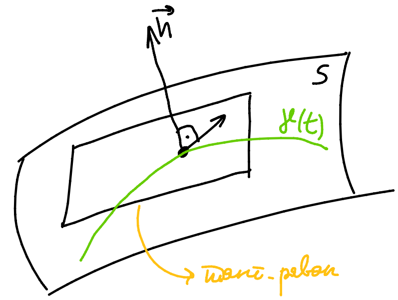
Посматрајмо КЛ: $\quad a(x,y,u)\frac{\partial u}{\partial x}+b(x,y,u)\frac{\partial u}{\partial y}=c(x,y,u)$ $\quad$ за $u(x,y)$

(ради једноставности $n=2$)

$S=\{(x,y,u(x,y))\}\subseteq\mathbb{R}^3$ → график решења $u(x,y)$ → површ у $\mathbb{R}^3$

$\vec{n}=(u_x(x,y),u_y(x,y),-1)$ – нормала на површ S у $(x,y,u(x,y))$

u решење $\Rightarrow$ $a(x,y,u)\frac{\partial u}{\partial x}+b(x,y,u)\frac{\partial u}{\dots}$



u решење $\Rightarrow \quad a(x,y,u)\dfrac{\partial u}{\partial x} + b(x,y,u)\dfrac{\partial u}{\partial y} = c(x,y,u)$

$\Rightarrow \quad a(x,y,u)\dfrac{\partial u}{\partial x} + b(x,y,u)\dfrac{\partial u}{\partial y} - c(x,y,u) = 0$

$\Rightarrow \quad \langle (a,b,c), \overset{\vec n}{(u_x, u_y, -1)} \rangle = 0$

$\Rightarrow \quad \langle (a,b,c), \vec n \rangle = 0$

$\Rightarrow \quad (a,b,c) \in$ тангентној равни на S

$$\Rightarrow \quad \left.\begin{aligned} x'(t) &= a(x,y,u) \\ y'(t) &= b(x,y,u) \\ u'(t) &= c(x,y,u) \end{aligned}\right\} \otimes \text{ – систем карактеристика}$$

$\gamma(t) = (x(t), y(t), u(t))$ – решење система $\otimes \Rightarrow \gamma(t) \in S$, $\forall t$

систем карakt. у сим. облику: $\dfrac{dx}{a(x,y,u)} = \dfrac{dy}{b(x,y,u)} = \dfrac{du}{c(x,y,u)} \left(= \dfrac{dt}{1}\right)$

уопштено: $\dfrac{dx_1}{a_1(x_1,\dots,x_n,u)} = \dfrac{dx_2}{a_2(x_1,\dots,x_n,u)} = \dots = \dfrac{dx_n}{a_n(x_1,\dots,x_n,u)} = \dfrac{du}{c(x_1,\dots,x_n,u)}$

$\Rightarrow$ n првих интеграла: $\psi_1, \dots, \psi_n$

$\boxed{T}$ ОР КЛ ПДЈ је $\Phi(\psi_1,\dots,\psi_n) = 0$, $\Phi \in C^1(U)$, $U \subseteq \mathbb{R}^n$

← инит. задато

За ХЛ ПДЈ: $a_1(x_1,\dots,x_n)\dfrac{\partial u}{\partial x_1} + \dots + a_n(x_1,\dots,x_n)\dfrac{\partial u}{\partial x_n} = 0$

систем карактеристика: $$\left.\begin{aligned} x_1'(t) &= a_1(x_1,\dots,x_n) \\ &\vdots \\ x_n'(t) &= a_n(x_1,\dots,x_n) \end{aligned}\right\}$$

$(u'(t) = 0 \Rightarrow u = const)$

у сим. облику: $\dfrac{dx_1}{a_1(x_1,\dots,x_n)} = \dots = \dfrac{dx_n}{a_n(x_1,\dots,x_n)} \Rightarrow \psi_1, \dots, \psi_{n-1}$ нез. 1. интеграла

$\boxed{T}$ ОР ХЛ ПДЈ: $u = \Phi(\psi_1,\dots,\psi_{n-1})$, $\Phi \in C^1(U)$, $U \subseteq \mathbb{R}^{n-1}$.

↑ екстр. задато

① $m,n,k\in\mathbb{R}\setminus\{0\}$. Решити ПДЈ:

$u(x,y,z)$ $\qquad (mz-ny)\cdot\frac{\partial u}{\partial x}+(nx-kz)\cdot\frac{\partial u}{\partial y}+(ky-mx)\cdot\frac{\partial u}{\partial z}=0 \rightsquigarrow$ ХЛ

$$\Downarrow$$

$$\frac{dx}{mz-ny}=\frac{dy}{nx-kz}=\frac{dz}{ky-mx} \qquad \psi_1,\psi_2=?$$

$$\frac{\alpha\,dx+\beta\,dy}{\alpha(mz-ny)+\beta(nx-kz)}=\frac{dz}{ky-mx}$$

уз z: $\alpha m-\beta k=0$ $\qquad \alpha=k,\ \beta=m \ \Rightarrow\ \frac{k\,dx+m\,dy}{n(-ky+mx)}=\frac{dz}{ky-mx}\ \Rightarrow\ k\,dx+m\,dy+n\,dz=0\ /\int$

$$\psi_1(x,y,z)=kx+my+nz=c_1$$

$$\frac{\alpha x\,dx+\beta y\,dy}{\alpha x(mz-ny)+\beta y(nx-kz)}=\frac{z\,dz}{z(ky-mx)}$$

уз xy: $-\alpha n+\beta n=0$ $\qquad \alpha=\beta=1\ \Rightarrow\ \frac{x\,dx+y\,dy}{mxz-kyz}=\frac{z\,dz}{z(ky-mx)}\ \Rightarrow\ x\,dx+y\,dy+z\,dz=0\ /\int$

$$\psi_2(x,y,z)=x^2+y^2+z^2=2c_2$$

ψ_1,ψ_2 нез? $\qquad \frac{D(\psi_1,\psi_2)}{D(x,y)}=\begin{vmatrix}k & m\\ 2x & 2y\end{vmatrix}=2(ky-mx)\neq 0$

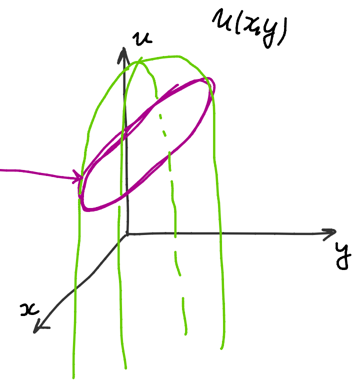
z - нез. пром.

$ky-mx\neq 0$

$$\Rightarrow \text{ОР}:\ u=\varphi(kx+my+nz,\ x^2+y^2+z^2),\ \varphi\in C^1(\mathbb{R}^2)$$

• Кошијев проблем: Одредити решење које садржи дату „површ“

$\varphi=?$



(2) Решити Кошијев проблем

$$x(z^2-y^2)\cdot\frac{\partial u}{\partial x} + y(x^2+z^2)\frac{\partial u}{\partial y} - z(x^2+y^2)\cdot\frac{\partial u}{\partial z} = 0, \qquad u\big|_{x=1} = u(1,y,z) = (y+z)^2.$$

ХЛ $\rightsquigarrow$ $$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2+z^2)} = \frac{dz}{-z(x^2+y^2)}$$

сличан зад. пре 2 часа: $$\left.\begin{aligned} &\psi_1(x,y,z) = x^2+y^2+z^2 \\ &\psi_2(x,y,z) = \frac{x}{yz} \quad \text{нез.} \end{aligned}\right\} \text{ОР: } u = \varphi(\psi_1,\psi_2),\ \varphi\in C^1(\mathbb{R}^2)$$

Кошијев услов

$u\big|_{x=1} = u(1,y,z) = (y+z)^2$. Тражимо $\varphi = ?$ тд. $u(1,y,z) = \varphi(\psi_1(1,y,z), \psi_2(1,y,z))$

$$y^2+z^2+2yz = (y+z)^2 = \varphi\left(1+y^2+z^2, \frac{1}{yz}\right)$$

$$\varphi\left(1+y^2+z^2, \frac{1}{yz}\right) = \boxed{(1+y^2+z^2)} - 1 + \frac{2}{\boxed{\frac{1}{yz}}}$$

$$\varphi(X,Y) = X - 1 + \frac{2}{Y} \checkmark$$

Кошијево решење: $u = \varphi(\psi_1,\psi_2) = \psi_1 - 1 + \frac{2}{\psi_2} = x^2+y^2+z^2-1+\frac{2yz}{x}$

(3) Решити Кошијев проблем $$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} + z^2 = 0, \qquad z=1, \quad xy = x+y$$

$z(x,y)$

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = \underbrace{-z^2}_{= c(x,y,z)}$$

(КЛ)

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

$\frac{dx}{x^2} = \frac{dy}{y^2} \;/\int$

$-\frac{1}{x} = -\frac{1}{y} + c_1$

$\frac{dy}{y^2} = \frac{dz}{-z^2} \;/\int$

$-\frac{1}{y} = \frac{1}{z} + c_2$

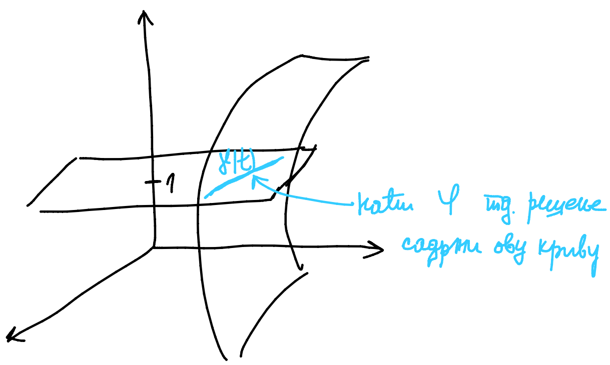
$\psi_1 = \frac{1}{x} - \frac{1}{y}$ $\psi_2 = \frac{1}{y} + \frac{1}{z}$

Нез: $\frac{D(\psi_1,\psi_2)}{D(x,z)} = \begin{vmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{z^2} \end{vmatrix} = \frac{1}{x^2z^2} \neq 0$

ОР: $\varphi\left(\frac{1}{x}-\frac{1}{y}, \frac{1}{y}+\frac{1}{z}\right) = 0$, $\varphi \in C^1(\mathbb{R}^2)$ имп. $z(x,y)$

Кошијев пр:

$z = 1$
$xy = x + y$



описујемо $\gamma(t)$ преко 1 параметра t:

$z = 1$

$xy = x + y$, $t = x$

$yt = t + y \Rightarrow y(t-1) = t$

$y = \frac{t}{t-1}$

$\gamma(t) = \left(t, \frac{t}{t-1}, 1\right)$

$$\varphi\left(\frac{1}{t} - \frac{1}{\frac{t}{t-1}}, \frac{1}{\frac{t}{t-1}} + \frac{1}{1}\right) = 0$$

$$\varphi\left(\frac{1-(t-1)}{t}, \frac{t-1}{t} + 1\right) = 0$$

$$\varphi\left(\frac{2-t}{t}, \frac{2t-1}{t}\right) = 0$$

$$\varphi\left(\frac{2}{t} - 1, 2 - \frac{1}{t}\right) = 0 \quad \longleftarrow \quad \varphi(X,Y) = X + 2Y - 3 \ \checkmark$$

$$\left(\frac{2}{t} - 1\right) + 2\left(2 - \frac{1}{t}\right) - 3 = 0$$

Кош. реш: $\psi_1 + 2\psi_2 - 3 = 0$

$$\frac{1}{x} - \frac{1}{y} + 2\left(\frac{1}{y} + \frac{1}{z}\right) - 3 = 0$$

имп. зад. z

може и експл.: $z = \frac{2}{3 - \left(\frac{1}{x} + \frac{1}{y}\right)}$

(4) Решити Кошијев проблем

$$x(x^2+3y^2)\frac{\partial z}{\partial x} + y(3x^2+y^2)\frac{\partial z}{\partial y} = 2z(x^2+y^2), \quad xy = z$$

(4) Решити Кошијев проблем

$$x(x^2+3y^2)\frac{\partial z}{\partial x} + y(3x^2+y^2)\frac{\partial z}{\partial y} = 2z(x^2+y^2) \quad , \quad \begin{array}{l} xy = z \\ x^2-y^2=z^2 \end{array}$$

$z(x,y) \rightsquigarrow$ (КЛ)

$$\frac{dx}{x(x^2+3y^2)} = \frac{dy}{y(3x^2+y^2)} = \frac{dz}{2z(x^2+y^2)} \qquad \psi_1, \psi_2 = ?$$

$$\frac{\frac{\alpha}{x}dx + \frac{\beta}{y}dy}{\alpha(x^2+3y^2)+\beta(3x^2+y^2)} = \frac{\frac{dz}{z}}{2(x^2+y^2)}$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$\alpha = \beta = \frac{1}{2}$:
$$\frac{\frac{dx}{2x} + \frac{dy}{2y}}{2(x^2+y^2)} = \frac{\frac{dz}{z}}{2(x^2+y^2)} \quad \Big/ \cdot 2(x^2+y^2) \Big/ \int$$

$$\frac{1}{2}\ln|x| + \frac{1}{2}\ln|y| = \ln|z| + c_1 \quad \cdots \quad \psi_1(x,y,z) = \frac{xy}{z^2}$$

идеја: $x^4-y^4 = (x^2-y^2)(\underline{\underline{x^2+y^2}})$

$$\frac{x\,dx - y\,dy}{x^2(x^2+3y^2)-y^2(3x^2+y^2)} = \frac{dz}{2z(x^2+y^2)}$$

$$\frac{x\,dx - y\,dy}{x^4-y^4} = \frac{dz}{2z(x^2+y^2)}$$

$$2\frac{x\,dx-y\,dy}{x^2-y^2} = \frac{dz}{z} \Rightarrow d(\ln|x^2-y^2|) = d(\ln|z|)$$

$$d(\ln|x^2-y^2|) = \frac{1}{x^2-y^2}\cdot 2x\,dx + \frac{1}{x^2-y^2}\cdot(-2y)\,dy = \frac{2x\,dx - 2y\,dy}{x^2-y^2}$$

$\int$

$$\ln|x^2-y^2| = \ln|z| + \tilde{c}$$

$\vdots$

$$\psi_2(x,y,z) = \frac{x^2-y^2}{z}$$

ψ_1, ψ_2 нез. → довати

ψ_1, ψ_2 нез. $\to$ домати

ОР: $\varphi(\psi_1, \psi_2) = 0, \quad \varphi \in C^1(\mathbb{R}^2)$

$$\left.\begin{array}{l} S_1:\ xy = z \\ S_2:\ x^2 - y^2 = z^2 \end{array}\right\}$$ у пресеку је крива c $\qquad c = S_1 \cap S_2$

$$\overline{\psi_1} = \psi_1\Big|_c = \left(\frac{xy}{z^2}\right)\Big|_c \underset{\substack{\uparrow \\ xy = z}}{=} \left(\frac{z}{z^2}\right)\Big|_c = \frac{1}{z}$$

циљ: изразити све преко 1 параметра
(ми немамо параметризацију од c)

$$\overline{\psi_2} = \psi_2\Big|_c = \left(\frac{x^2 - y^2}{z}\right)\Big|_c \underset{\substack{\uparrow \\ x^2 - y^2 = z^2}}{=} \left(\frac{z^2}{z}\right)\Big|_c = z$$

$$\varphi(\overline{\psi_1}, \overline{\psi_2}) = 0 \Rightarrow \varphi\left(\frac{1}{z}, z\right) = 0 \leftsquigarrow \varphi(X, Y) = X \cdot Y - 1 \ \checkmark$$

Кош. реш: $\psi_1 \cdot \psi_2 - 1 = 0 \Rightarrow \dfrac{xy}{z^2} \cdot \dfrac{x^2 - y^2}{z} = 1 \Rightarrow \ldots \ z = \sqrt[3]{xy(x^2 - y^2)}$

може чак и експл.