11 - уторак 10.05.2022.

доказати: $x_1' = x_2 - x_1 x_2^2$

$x_2' = -x_1^3$

а) $V(x_1,x_2) = ax_1^2 + bx_2^2$ се не може користити као ф-ја Љапунова

б) $V(x_1,x_2) = ax_1^4 + bx_2^2 \leftarrow$ потражити у овом облику

6. Испитати стабилност положаја равнотеже $X^* = (0,0)$ динамичког система

$$x_1' = -\sin x_2$$
$$x_2' = x_1.$$

$A = dF(0,0) \rightarrow$ не даје одговор

$V = ?$, $F(x_1,x_2) = (-\sin x_2, x_1)$

циљано: да буде стабилан (али не асимптотски)

хоћемо $\dot{V}(X) = 0 = \langle \underbrace{\nabla V(X)}_{(v_1,v_2)}, F(X)\rangle = \underbrace{v_1}_{x_1} \cdot (-\sin x_2) + \underbrace{v_2}_{\sin x_2} \cdot x_1 = 0$

Хоћемо: $\nabla V(X) = (x_1, \sin x_2) \rightsquigarrow V = ?$

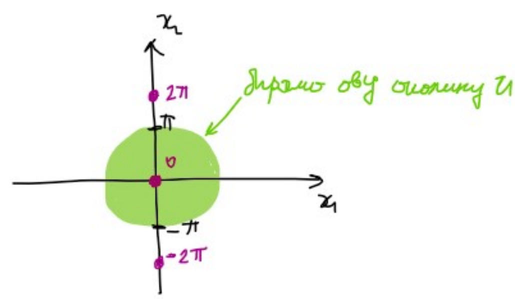
$V(x_1,x_2) = \dfrac{x_1^2}{2} - \cos x_2 + C \rightsquigarrow$ позитивно дефинитна

$V(0,0) = 0 \longrightarrow C = 1$

$V(X) > 0$, $X \neq (0,0)$

$V(X) = \underbrace{\dfrac{x_1^2}{2}}_{\geq 0} + \underbrace{1 - \cos x_2}_{\geq 0} > 0$, $X \in U \setminus \{(0,0)\}$

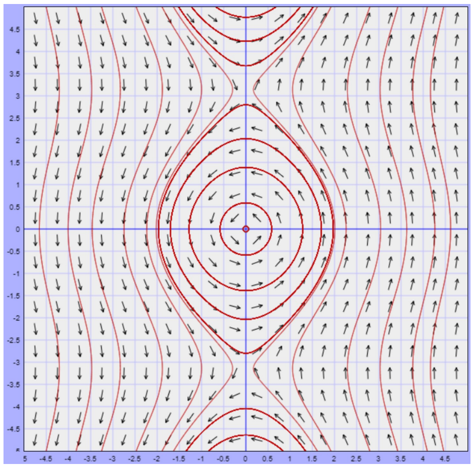
када су $\frac{x_1^2}{2} = 1 - \cos x_2 = 0$? $x_1 = 0$ и $x_2 = 2k\pi, k \in \mathbb{Z}$

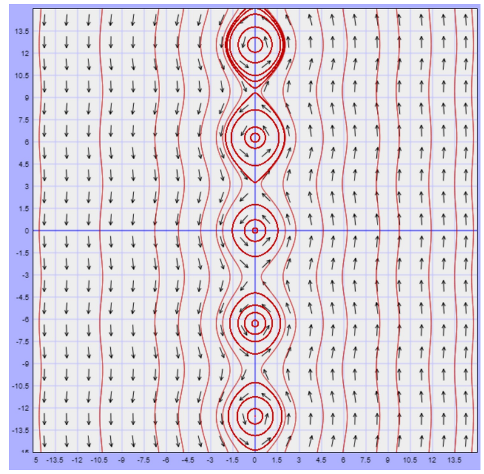


бирамо нпр. $U = B((0,0); \pi)$

1) $V \in C^1(\mathbb{R}^2)$
2) $V(0,0) = 0$, $V(X) > 0$, $X \in U \setminus \{(0,0)\}$
3) $\dot{V}(X) = \langle \nabla V(X), F(X) \rangle = 0 \leq 0$

$\Rightarrow X^* = (0,0)$ стабилан еквилибријум





Системи у симетричном облику

$$\frac{dx_1}{f_1(x_1,\dots,x_n)} = \frac{dx_2}{f_2(x_1,\dots,x_n)} = \dots = \frac{dx_n}{f_n(x_1,\dots,x_n)}$$

← систем у симетричном облику реда n

$$\Updownarrow$$

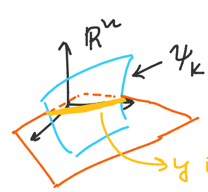
$$\left.\begin{aligned}
x_1' &= \frac{dx_1}{dx_n} = \frac{f_1(x_1,\dots,x_n)}{f_n(x_1,\dots,x_n)} \\
x_2' &= \frac{dx_2}{dx_n} = \frac{f_2}{f_n} \\
&\vdots \\
x_{n-1}' &= \frac{dx_{n-1}}{dx_n} = \frac{f_{n-1}}{f_n}
\end{aligned}\right\}$$

← систем у нормалном облику реда $n-1$

- Изабрали смо x_n за независну променљиву → захтевамо $f_n \neq 0$
- Имамо слободу да бирамо независну променљиву
- Први интеграл је ф-ја која је константна дуж решења

$\downarrow$ ψ_i

$$\psi_1(x_1,\dots,x_n) = c_1$$
$$\psi_2(x_1,\dots,x_n) = c_2$$
$$\vdots$$
$$\psi_{n-1}(x_1,\dots,x_n) = c_{n-1}$$



„лепо" се секу те хипертравни

• Први интеграли $\psi_1, \dots, \psi_{n-1}$ су линеарно независни ако је

$$\frac{D(\psi_1, \dots, \psi_{n-1})}{D(\underbrace{x_1, \dots, x_{n-1}}_{\text{изоставимо нез. пром.}})} \neq 0$$

• Опште решење: $(\psi_1, \dots, \psi_{n-1}) = const \in \mathbb{R}^{n-1}$

① Решити системе у симетричном облику:

а) $\dfrac{dx}{x+y^2+z^2} = \dfrac{dy}{y} = \dfrac{dz}{z}$

б) $\dfrac{dx}{y-u} = \dfrac{dy}{z-x} = \dfrac{dz}{u-y} = \dfrac{du}{x-z}$

в) $\dfrac{dx}{4y-3z} = \dfrac{dy}{4x-2z} = \dfrac{dz}{2y-3x}$

г) $\dfrac{dx}{x^2-y^2-z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$

а) независна: z, захтевамо $z \neq 0$ $\left(\dfrac{dz}{z}\right)$

$$\frac{dx}{dz} = \frac{x+y^2+z^2}{z}$$

$$\frac{dy}{dz} = \frac{y}{z} \rightsquigarrow \frac{dy}{y} = \frac{dz}{z} \Big/ \int$$

$$\ln|y| = \ln|z| + c_1 \;/ e^{\square}$$

$$|y| = e^{c_1} \cdot |z|$$

$$\Rightarrow y = c \cdot z \quad , c \in \mathbb{R}$$

$$\Rightarrow \frac{y}{z} = c \rightsquigarrow const \in \mathbb{R} \quad \psi_1$$

$$\psi_1(x,y,z) = \frac{y}{z}$$

$$x_z' = \frac{x+y^2+z^2}{z} \underset{y=cz}{=} \frac{x+c^2z^2+z^2}{z} = \frac{x}{z} + z\cdot(c^2+1) \rightarrow \text{линеарна } x(z)$$

$$x_z' - \frac{x}{z} = z(c^2+1)$$

$$p(z) = -\frac{1}{z}$$

$$q(z) = z(c^2+1)$$

$$\overset{x}{\overset{\parallel}{x(z)}} = C_2 \cdot z + (1+c^2)z^2 = C_2 \cdot z + \left(1+\left(\frac{y}{z}\right)^2\right)\cdot z^2 = C_2 z + z^2 + y^2$$

$$\Rightarrow C_2 = \frac{x - y^2 - z^2}{z} = \psi_2(x,y,z)$$

$$\frac{D(\psi_1,\psi_2)}{D(x,y)} = \begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{z} \\ \frac{1}{z} & -\frac{2y}{z} \end{vmatrix} = -\frac{1}{z^2} \neq 0$$

↳ без z - независна

$\Rightarrow \psi_1$ и ψ_2 лин. нез. $\Rightarrow$ ОР: $\left.\begin{matrix} \psi_1 = C_1 \\ \psi_2 = C_2 \end{matrix}\right\}$ $\quad C_1, C_2 \in \mathbb{R}$

б) $$\frac{dx}{y-u} = \frac{dy}{z-x} = \frac{dz}{u-y} = \frac{du}{x-z}$$ ⇝ 3 прва интеграла

$$\frac{dx}{y-u} = \frac{dz}{u-y} \quad /\cdot(u-y)$$

$$-dx = dz$$

$$dx + dz = 0 \quad /\int$$

$$x + z = C_1$$

$$\psi_1(x,y,z,t) = x + z$$

$$\boxed{\int dx = x + c}$$

$$\frac{dy}{z-x} = \frac{du}{x-z} \quad /\cdot(x-z)$$

$$-dy = du$$

$$dy + du = 0 \quad /\int$$

$$y + u = C_2$$

$$\psi_2(x,y,z,u) = y + u$$

$$\frac{dx}{y-u} + \frac{dz}{u-y} = \frac{dy}{z-x} + \frac{du}{x-z}$$

$$\frac{dx - dz}{y-u} = \frac{du - dy}{x-z}$$

$$(x-z)\,d(x-z) = (y-u)\,d(u-y)$$

$$(x-z)d(x-z) + (u-y)d(u-y) = 0 \quad /\int$$

$$\frac{(x-z)^2}{2} + \frac{(u-y)^2}{2} = C_3$$

$$(x-z)^2 + (u-y)^2 = 2C_3 = C_4$$

$$\psi_3(x,y,z,u) = (x-z)^2 + (u-y)^2$$

$$\boxed{\begin{matrix} d(f \pm g) = df \pm dg \\ \int A\,dA = \frac{A^2}{2} + c \end{matrix}}$$

Бирамо независну променљиву (произвољно). Нпр. $\underline{y} \Rightarrow z - x \neq 0$

$$\frac{D(\psi_1,\psi_2,\psi_3)}{D(x,z,u)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2(x-z) & 2(z-x) & 2(u-y) \end{vmatrix} = 1\cdot(-1)\cdot\begin{vmatrix} 1 & 1 \\ 2(x-z) & 2(z-x) \end{vmatrix} = 2(x-z)-2(z-x) = 4(x-z) \neq 0$$

$$\Rightarrow \text{ОР}: \ (\psi_1,\psi_2,\psi_3) = c \in \mathbb{R}^3$$

в) $\dfrac{dx}{4y-3z} = \dfrac{dy}{4x-2z} = \dfrac{dz}{2y-3x}$

метод линеарних комбинација

идеја: $\dfrac{dx}{f_1} = \dfrac{dy}{f_2} = \dfrac{dz}{f_3}$

$\alpha,\beta\in\mathbb{R}$

$$\frac{\alpha dx+\beta dy}{\alpha f_1+\beta f_2} = \frac{dz}{f_3}$$

$\alpha,\beta = ?$

$$\frac{\alpha dx + \beta dy}{\alpha(4y-3z)+\beta(4x-2z)} = \frac{dz}{2y-3x}$$

$$\frac{\alpha dx+\beta dy}{x\cdot(4\beta)+y(4\alpha)+\underbrace{\boxed{z(-3\alpha-2\beta)}}_{=0}} = \frac{dz}{2y-3x}$$

$\dfrac{4\beta}{4\alpha} = \dfrac{-3}{2}$, $-3\alpha-2\beta=0$

$\dfrac{\beta}{\alpha} = -\dfrac{3}{2}$, $\dfrac{\beta}{\alpha} = -\dfrac{3}{2}$

нпр. $\beta=3,\ \alpha=-2$

$$\Rightarrow \frac{-2dx+3dy}{\underbrace{12x-8y}_{=-4\cdot(2y-3x)}} = \frac{dz}{2y-3x} \ \Big/\cdot(12x-8y)$$

$$\Downarrow$$

$$-2dx+3dy = -4dz$$

$$-2dx+3dy+4dz = 0 \ \Big/\int$$

$$-2x+3y+4z = c_1$$

$$\psi_1(x,y,z) = -2x+3y+4z$$

Како год да покушамо на исту фору $\rightsquigarrow$ добили бисмо ψ_2 које је зависно са ψ_1

$$\frac{\alpha x\cdot dx + \beta y\cdot dy}{\alpha x\cdot(4y-3z)+\beta y\cdot(4x-2z)} = \frac{z dz}{z(2y-3x)}$$

$$\frac{\alpha x\, dx+\beta y\, dy}{\underbrace{\boxed{xy(4\alpha+4\beta)}}_{=0}+yz(-2\beta)+zx(-3\alpha)} = \frac{z\,dz}{2yz-3xz}$$

$\to 4\alpha+4\beta=0 \Rightarrow \alpha=-\beta$

$\to \dfrac{-2\beta}{-3\alpha} = \dfrac{2}{-3} \Rightarrow \alpha=-\beta$

нпр. $\alpha=1, \beta=-1$

$$\frac{x\,dx - y\,dy}{\cancel{2yz-3xz}} = \frac{z\,dz}{\cancel{2yz-3xz}}$$

$$x\,dx - y\,dy - z\,dz = 0 \;/\int$$

$$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = c_2$$

$$\psi_2(x,y,z) = x^2 - y^2 - z^2$$

нпр. x независна
$\Downarrow$
$4y-3z\neq 0$

$$\frac{D(\psi_1,\psi_2)}{D(y,z)} = \begin{vmatrix} 3 & 4 \\ -2y & -2z \end{vmatrix} = -6z + 8y = 2(4y-3z) \neq 0$$

ОР: $\psi_1 = c_1$, $\psi_2 = c_2$ $\quad c_1, c_2 \in \mathbb{R}$

г) $$\frac{dx}{x^2-y^2-z^2} = \underbrace{\frac{dy}{2xy} = \frac{dz}{2xz}}$$

$$\frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z}$$

$$\frac{dy}{y} = \frac{dz}{z} \;/\int \;\rightsquigarrow\; y = c\cdot z, \quad c\in\mathbb{R}$$

$$\psi_1(x,y,z) = \frac{y}{z}$$

z-независна, $2xz \neq 0$

$$x'_z = \frac{dx}{dz} = \frac{x^2-y^2-z^2}{2xz} = \frac{x^2-c^2z^2-z^2}{2xz}$$

$$x'_z - \frac{x}{2z} = -\frac{(c^2+1)z}{2x} \xrightarrow{\text{Бернулијева}} x'_z - p(z)\cdot x = g(z)\cdot x^{\alpha}$$

$\alpha = -1$

$$u(z) = (x(z))^{1-\alpha} = (x(z))^2 \ldots$$

домаћи: решити Бер. јну, $\psi_2 = \frac{x^2+y^2+z^2}{z}$, проверити независност ψ_1 и ψ_2, ОР

(2) Решити систем

$$\begin{cases} \dfrac{dy}{dx} = y' = \dfrac{y(x^2+z^2)}{x(z^2-y^2)} \\[2ex] \dfrac{dz}{dx} = z' = \dfrac{z(x^2+y^2)}{x(y^2-z^2)} \end{cases}$$

идеја: превести у симетричан облик

$$\frac{dx}{\ldots} = \frac{dy}{\quad} = \frac{dz}{\quad}$$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(x^2+z^2)} = \frac{dz}{z(x^2+y^2)}$$

$$\frac{\alpha x \cdot dx + \beta y\, dy}{\alpha x^2(y^2-z^2) - \beta y^2(x^2+z^2)} = \frac{z\, dz}{z^2(x^2+y^2)}$$

уз x^2y^2 то $=0$ $\Rightarrow \alpha - \beta = 0 \Rightarrow$ нпр. $\alpha = \beta = 1$

$$\frac{x\,dx + y\,dy}{-x^2z^2 - y^2z^2} = \frac{z\,dz}{z^2x^2 + z^2y^2} \Rightarrow x\,dx + y\,dy = -z\,dz \Big/ \int$$

$$\frac{x^2+y^2+z^2}{2} = c_1$$

$$\psi_1(x,y,z) = x^2+y^2+z^2$$

$$\frac{\frac{\alpha}{x}dx + \frac{\beta}{y}dy}{\frac{\alpha}{\cancel{x}}\cdot\cancel{x}(y^2-z^2) - \frac{\beta}{\cancel{y}}\cdot\cancel{y}(x^2+z^2)} = \frac{\frac{dz}{z}}{\frac{1}{\cancel{z}}\cdot\cancel{z}(x^2+y^2)}$$

$$\frac{\frac{\alpha dx}{x} + \frac{\beta dy}{y}}{\alpha y^2 - \beta x^2 + z^2(-\alpha-\beta)} = \frac{\frac{dz}{z}}{x^2+y^2}$$

$$\boxed{\int \frac{dx}{x} = \ln|x| + c}$$

$$\alpha = 1, \beta = -1: \quad \frac{\frac{dx}{x} - \frac{dy}{y}}{\cancel{y^2+x^2}} = \frac{\frac{dz}{z}}{\cancel{y^2+x^2}}$$

$$\frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z} = 0 \Big/ \int$$

$$\ln|x| - \ln|y| - \ln|z| = c_2$$

$$\ln\left|\frac{x}{yz}\right| = c_2$$

$$\vdots$$

$$\frac{x}{yz} = c_3, \quad \psi_2(x,y,z) = \frac{x}{yz}$$

доказати: ψ_1 и ψ_2 независни

Присетити се и: $d(xy) = x\,dy + y\,dx$

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$$

никако: $\int x\,dy \overset{?}{=} xy + c$

или слично