1. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

(1) $A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$; (2) $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$; (3) $A = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}$.

(2) $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \rightsquigarrow \lambda_1 = 3,\ \lambda_2 = 2$

$$\left.\begin{aligned} x_1' &= 4x_1 - 2x_2 \\ x_2' &= x_1 + x_2 \end{aligned}\right\} \quad x_1' = x_2' = 0 \Rightarrow 4x_1 - 2x_2 = x_1 + x_2 = 0 \Rightarrow x_1 = x_2 = 0 \Rightarrow X^* = (0,0)$$

$\lambda_1 = 3 \rightsquigarrow \gamma_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $\lambda_2 = 2 \rightsquigarrow \gamma_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

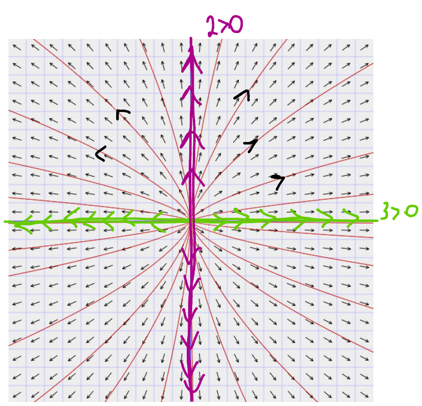
$A \sim \dot{J} = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix}$

ОР: $X(t) = c_1 e^{3t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $c_1, c_2 \in \mathbb{R}$

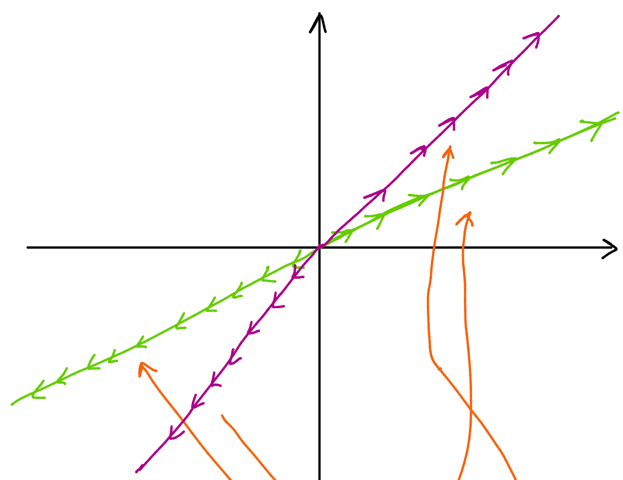
https://aeb019.hosted.uark.edu/pplane.html

λ_1, λ_2 позитивне (реалне) и различите $\longrightarrow$ нестабилан чвор

$\dot{J} = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix}$



$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$



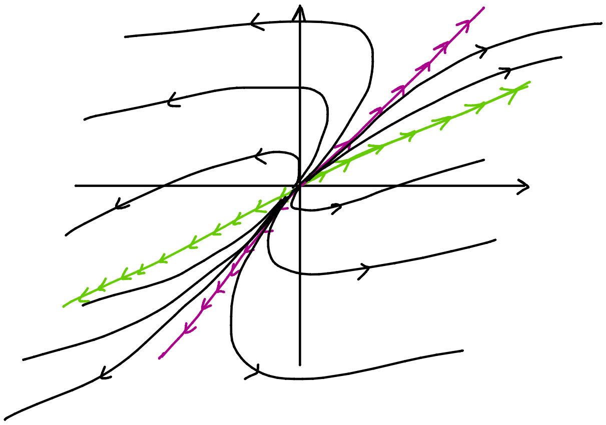
$c_1 = 1$, $c_2 = 0$: $X(t) = e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{3t} \\ e^{3t} \end{bmatrix}$ $2x_2 = x_1,\ x_1 > 0$; $x_2 = \frac{x_1}{2}$

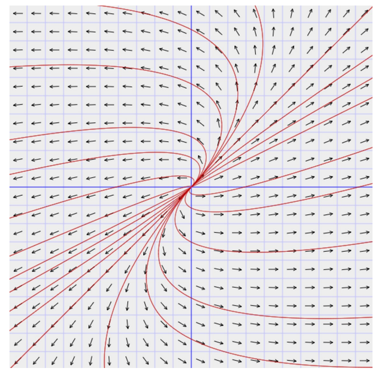
$c_1 = -1$, $c_2 = 0$: $X(t) = -e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2e^{3t} \\ -e^{3t} \end{bmatrix}$ $x_1 = 2x_2,\ x_1 < 0$

$c_1 = 0$, $c_2 = 1$: $X(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$ $x_1 = x_2 > 0$

$c_1 = 0$
$c_2 = 1$ $\quad X(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$ $\quad x_1 = x_2 > 0$

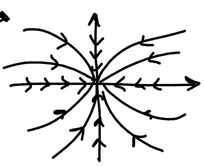
$c_1 = 0$
$c_2 = -1$ $\quad X(t) = \begin{bmatrix} -e^{2t} \\ -e^{2t} \end{bmatrix}$ $\quad x_1 = x_2 < 0$



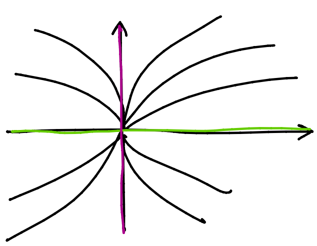


$\lambda_1, \lambda_2 < 0, \lambda_1 \neq \lambda_2$

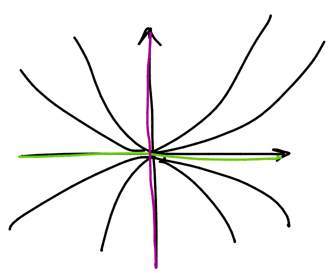
J →



стабилан чвор



$|\lambda_2| < |\lambda_1|$

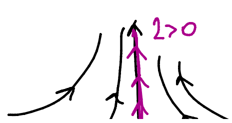


$|\lambda_1| < |\lambda_2|$

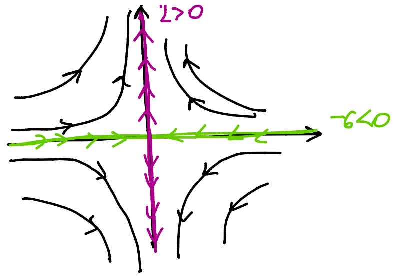
(3) $A = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}$

$-x_1 + 3x_2 = 5x_1 - 3x_2 = 0 \rightsquigarrow X^* = (0,0)$

$\lambda_1 = -6$
$\lambda_2 = 2$ реалне и разл. знака → седло



$$\dot{J} = \begin{bmatrix} -6 & \\ & 2 \end{bmatrix}$$



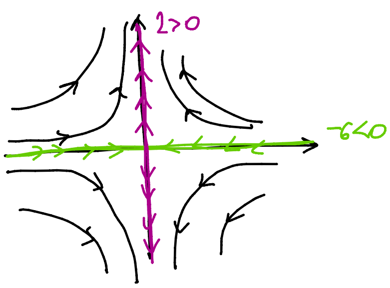
$$\lambda_1 = -6 \rightsquigarrow \gamma_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

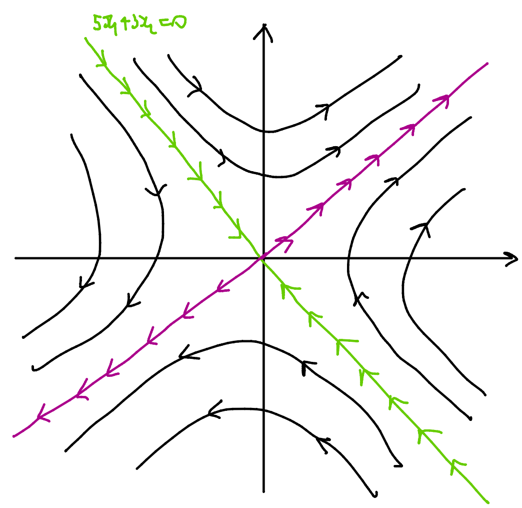
$$\lambda_2 = 2 \rightsquigarrow \gamma_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X(t) = c_1 e^{-6t} \begin{bmatrix} 3 \\ -5 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , c_1, c_2 \in \mathbb{R}$$

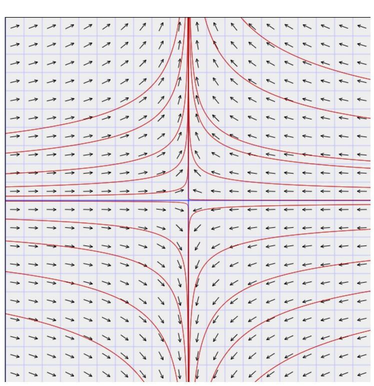
$c_1 = \pm 1$, $c_2 = 0$: $\quad X(t) = \pm \begin{bmatrix} 3e^{-6t} \\ -5e^{-6t} \end{bmatrix} \quad 5x_1 + 3x_2 = 0$

$c_1 = 0$, $c_2 = \pm 1$: $\quad X(t) = \pm \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} \quad x_1 = x_2$

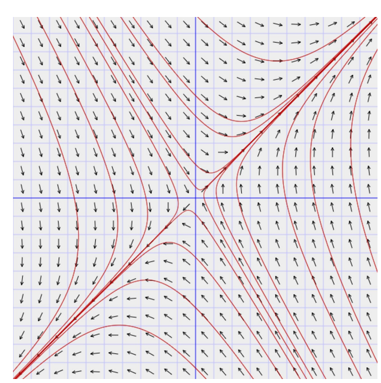


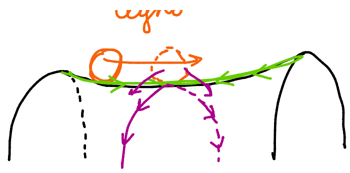


$\dot{J}$ ⤵



A ⤵





2. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

(1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; (2) $A = \begin{bmatrix} -2 & -5 \\ 2 & 2 \end{bmatrix}$;

(3) $A = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}$. (4) $A = \begin{bmatrix} 7 & -10 \\ 4 & -5 \end{bmatrix}$.

(1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\hookrightarrow \lambda_{1/2} = \pm i$

$X^* = (0,0)$

$Re(\lambda_{1/2}) = 0$ – центар

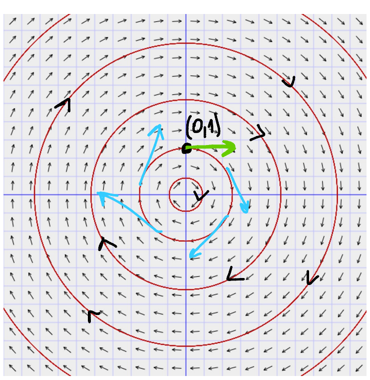
$x_1(t)$

$$X_1(t) = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

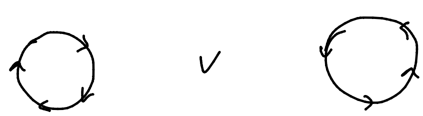
$x_2(t)$

испоставе се: $x_1(t)^2 + x_2(t)^2 = c_1^2 + c_2^2 = const$

$\hookrightarrow$ кругови (периодичне)



смер?



∨

$(0,1) \rightsquigarrow$ како изгледа извод?

$$X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad X' = AX = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(2) $A = \begin{bmatrix} -2 & -5 \\ 2 & 2 \end{bmatrix} \rightsquigarrow \lambda_{1/2} = \pm i\sqrt{6}$

$X^* = (0,0)$

$Re(\lambda_{1/2}) = 0 \rightarrow$ центар

$$(A - i\sqrt{6}E)\gamma = 0 \; : \; \gamma = \begin{bmatrix} 5 \\ -2-i\sqrt{6} \end{bmatrix}$$

$$\psi(t) = e^{i\sqrt{6}t} \cdot \begin{bmatrix} 5 \\ -2-i\sqrt{6} \end{bmatrix} = \left(\cos(\sqrt{6}t) + i\sin(\sqrt{6}t)\right) \begin{bmatrix} 5 \\ -2-i\sqrt{6} \end{bmatrix} \begin{matrix} \rightsquigarrow Re \\ \rightsquigarrow Im \end{matrix}$$

$$X(t) = c_1 \cdot \begin{bmatrix} 5\cos(\sqrt{6}t) \\ -2\cos(\sqrt{6}t)+\sqrt{6}\sin(\sqrt{6}t) \end{bmatrix} + c_2 \cdot \begin{bmatrix} 5\sin(\sqrt{6}t) \\ -2\sin(\sqrt{6}t)-\sqrt{6}\cos(\sqrt{6}t) \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

↳ тешко нацртати

$$A = T \cdot J \cdot T^{-1}$$

$$X' = AX \longrightarrow T^{-1}X' = J\,T^{-1}X \qquad J = \begin{bmatrix} 0 & \sqrt{6} \\ -\sqrt{6} & 0 \end{bmatrix}$$

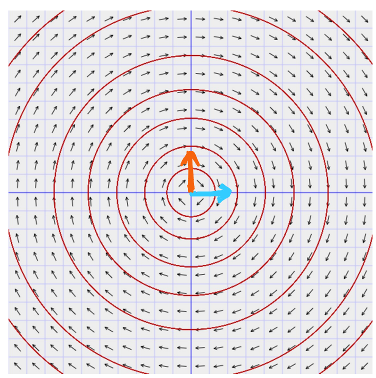
$$y = T^{-1} \cdot x$$

$$y' = T^{-1} \cdot x'$$

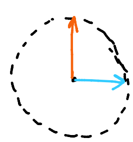
$$y' = Jy$$

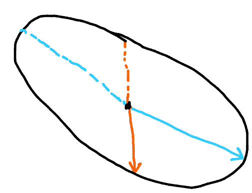
→ знамо да су кругови

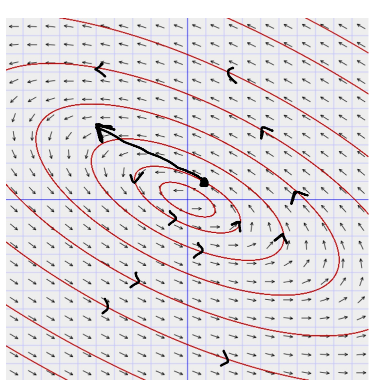
$$J = \begin{bmatrix} 0 & \sqrt{6} \\ -\sqrt{6} & 0 \end{bmatrix} \rightsquigarrow$$



$$T = \begin{bmatrix} \mathrm{Re}\, v & \mathrm{Im}\, v \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -2 & -\sqrt{6} \end{bmatrix}$$







Пример:

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad X' = AX = \begin{bmatrix} -2 & -5 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

(4) $A = \begin{bmatrix} 7 & -10 \\ 4 & -5 \end{bmatrix}$

$X^* = (0,0)$ $\hookrightarrow \lambda_{1/2} = 1 \pm 2i$

$Re(\lambda_{1/2}) \neq 0$ спирала (фокус)

код нас $(1>0)$

$Re(\lambda_{1/2}) > 0$: нестабилна спирала = ФТ иду од X^*

$Re(\lambda_{1/2}) < 0$: стабилна спирала = ФТ иду ка X^*

$\underline{\underline{1+2i}}$

$A = T \cdot J \cdot T^{-1}$

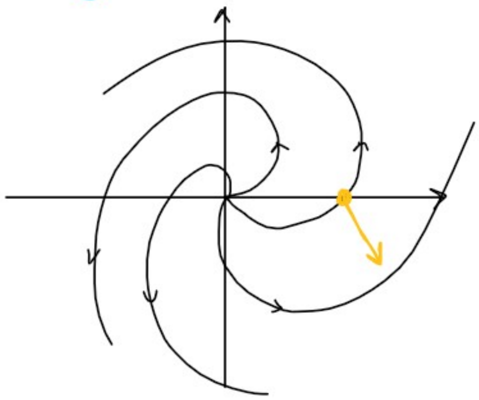
$(A - (1+2i)E)\gamma = 0$

$J = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

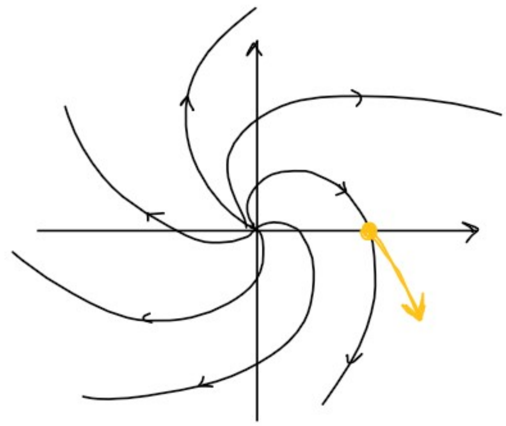
$\therefore \gamma = \begin{bmatrix} 5 \\ 3-i \end{bmatrix}$

$T = \begin{bmatrix} 5 & 0 \\ 3 & -1 \end{bmatrix}$

$J \rightsquigarrow$

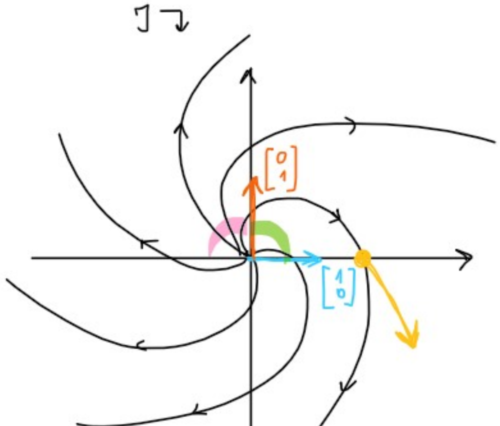


V



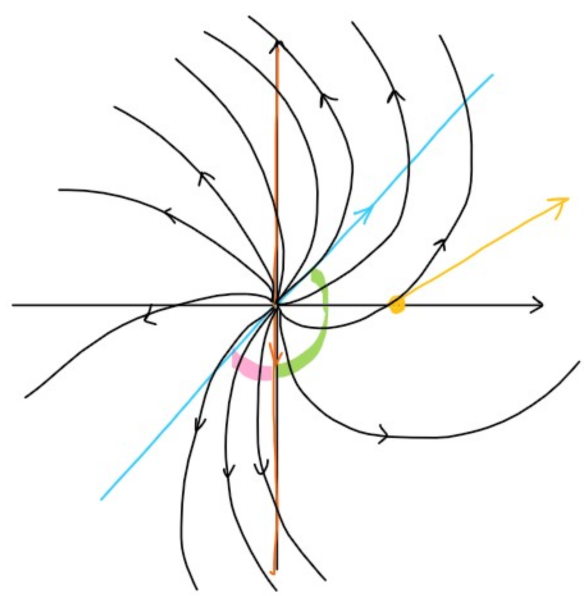
која је наша спирала? смер! $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow X' = JX = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow$ десна наша

J ↴



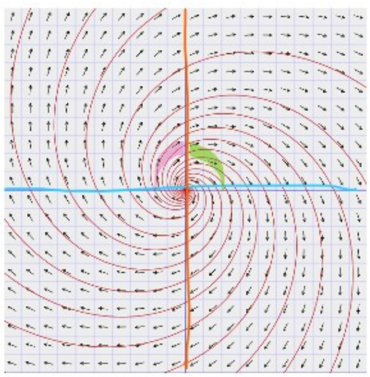
$\rightsquigarrow \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$\rightsquigarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

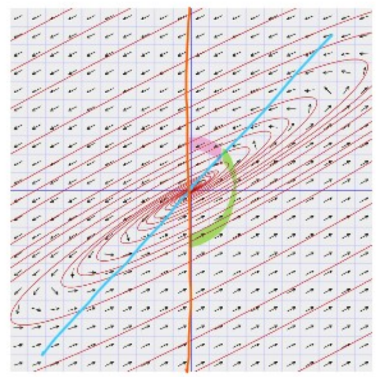


смер: $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $X' = AX = \begin{bmatrix} 7 & -10 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

J



A



3. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

(1) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$;

(2) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$;

(3) $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$;

(4) $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$;

(2) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\qquad \lambda_1 = \lambda_2 = -1$

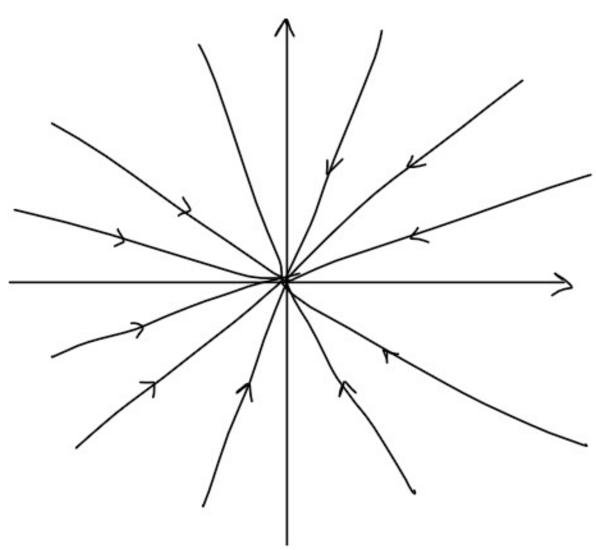
$X^* = (0,0)$ $\qquad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \vee \cancel{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}}$ $\qquad \lambda \neq 0$

$\lambda > 0$: нестабилна звезда

$\lambda < 0$: стабилна звезда

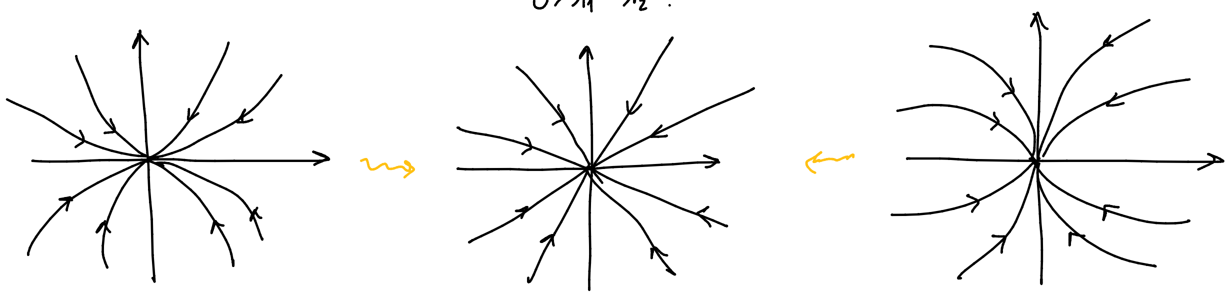
код нас $(-1<0)$

звезда (сингуларни чвор)



$\dot{J} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$:

$0 > \lambda_1 > \lambda_2$: $0 > \lambda_1 = \lambda_2$: $0 > \lambda_2 > \lambda_1$:



(4) $A = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix}$

$\lambda_1 = \lambda_2 = -2$

$X^* = (0,0)$

Жордан: $J = \begin{bmatrix} -2 & \\ & -2 \end{bmatrix}$ v $\dot{J} = \begin{bmatrix} -2 & 1 \\ & -2 \end{bmatrix}$?

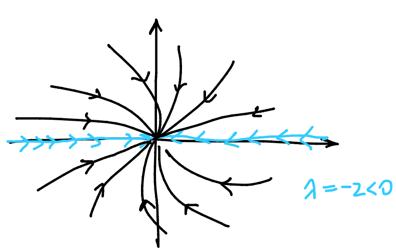
$(A+2E)\gamma_1 = 0$, $\gamma_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\Rightarrow$ 1 Жорд. блок $\Rightarrow$ $\dot{J} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

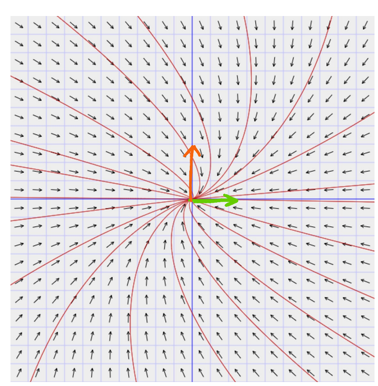
$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $\rightsquigarrow$ дегенерисани чвор

$\lambda \neq 0$ $\lambda > 0$: нестабилан

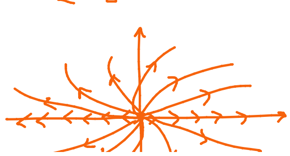
$\lambda < 0$: стабилан   код нас

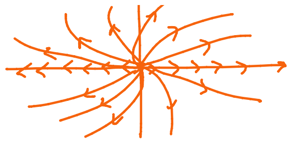
$\dot{J} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ $\rightsquigarrow$

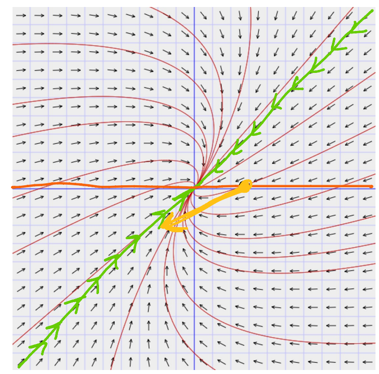




У нестабилном случају $\dot{J} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\lambda > 0$:







уопштени

$(A+2E)\gamma_2 = \gamma_1 \quad \ldots \quad \gamma_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

смер: $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X' = AX = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$