① $y' = Ay$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$$

$$k = 4$$

$$(A-2E)\gamma = 0 \Rightarrow \dots \quad \gamma \in \mathcal{L}in\Big\{ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\gamma_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\gamma_2} \Big\}^{\rightsquigarrow \dim = 2} \Rightarrow n = 2 \Rightarrow 2 \text{ Ж. блока}$$

$$\varphi(\lambda) = (\lambda - 2)^4$$

$$\mu(\lambda) = (\lambda-2)^k, \quad 1 \le k \le 4$$

$\mu(A) = 0:$ $\quad A - 2E \neq 0$

$(A-2E)^2 \neq 0$

$(A-2E)^3 = 0 \Rightarrow \deg \mu = 3 \Rightarrow$ највећи ред Ж. блока $3 \Rightarrow J = \begin{bmatrix} \boxed{2} & & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix}$

γ_1, γ_2 – сопствени $\rightsquigarrow$ још 2 уопштена

уопштени за γ_2:

$$(A-2E)\gamma = \gamma_2 \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$0 = 0$

$0 = 1$ ↯ нема решења $\Rightarrow \gamma_2$ нема уопштени

$d = 0$

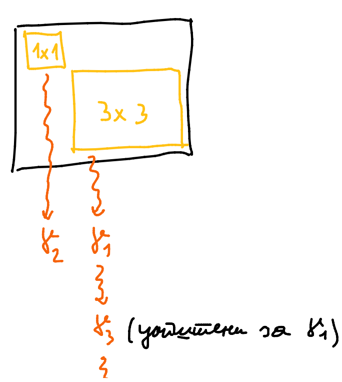
$a = 0$

уопштени за γ_1:

$$(A-2E)\gamma = \gamma_1$$

$$\begin{matrix} 0 = 0 \\ 0 = 0 \\ d = 1 \\ a = 0 \end{matrix} \rightsquigarrow \gamma = \begin{bmatrix} 0 \\ b \\ c \\ 1 \end{bmatrix} \xrightarrow{b = c = 0} \gamma_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(узмемо 1 уопштени само)



$\leadsto$ γ_4 (присъединени за γ_3)

присъединени за γ_3:

$$(A-2E)\gamma = \gamma_3$$

$$\begin{aligned} 0 &= 0 \\ 0 &= 0 \\ d &= 0 \\ a &= 1 \end{aligned} \leadsto \gamma = \begin{bmatrix} 1 \\ b \\ c \\ 0 \end{bmatrix} \xrightarrow{b=c=0} \gamma_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{J} = \begin{bmatrix} \boxed{2} & & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix}$$

$$T = \begin{bmatrix} \gamma_2 & \gamma_1 & \gamma_3 & \gamma_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$e^{x\dot{J}} = ? \qquad \dot{J} = \begin{bmatrix} 2 & \\ & B \end{bmatrix} \Rightarrow e^{x\dot{J}} = \begin{bmatrix} e^{2x} & \\ & e^{xB} \end{bmatrix} = \begin{bmatrix} e^{2x} & 0 & 0 & 0 \\ 0 & e^{2x} & xe^{2x} & \frac{x^2}{2}e^{2x} \\ 0 & 0 & e^{2x} & xe^{2x} \\ 0 & 0 & 0 & e^{2x} \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix} = \underset{D}{\underset{\parallel}{\begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}}} + \underset{N}{\underset{\parallel}{\begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}}} \qquad N^2 = \begin{bmatrix} & & 1 \\ & & \\ & & \end{bmatrix}$$

$$N^3 = 0, \quad N^k = 0, \ k \geq 3$$

$$DN = ND \overset{②}{\Rightarrow} e^{xB} = e^{xD} \cdot e^{xN} = e^{2x} \cdot E \cdot \left(E + xN + \frac{x^2}{2}N^2\right) = e^{2x} \cdot \begin{bmatrix} 1 & x & \frac{x^2}{2} \\ & 1 & x \\ & & 1 \end{bmatrix}$$

$$\text{OP}: \ y(x) = T \cdot e^{x\dot{J}} \cdot c, \ c \in \mathbb{R}^4$$

② $y' = Ay$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\varphi(\lambda) = (\lambda-2)^3$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 2, \ k = 3$$

$$(A-2E)\gamma = 0 \Rightarrow \ldots \ \gamma = a \cdot \underset{\gamma_1}{\underset{\parallel}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}} + b \cdot \underset{\gamma_2}{\underset{\parallel}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}} \Rightarrow m = \dim\ker(A-2E) = 2 \Rightarrow 2 \text{ блока}$$

$$\dot{J} = \begin{bmatrix} 2 & 1 & \end{bmatrix}$$

једина могућност: $J = \begin{array}{c} \quad\;\; \gamma_1 \quad \gamma_2 \\ \begin{bmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{bmatrix} \end{array}$

$$e^{xJ} = \begin{bmatrix} e^{x\begin{bmatrix}2&1\\&2\end{bmatrix}} & \\ & e^{2x} \end{bmatrix} = \dots = e^{2x}\begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T = ?$ – имамо 2 сопствена, треба нам 1 уопштени – да ли одговара γ_1 или γ_2?

$(A-2E)\gamma_3 = \gamma_1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$b+c=1$
$-b-c=0$
$b+c=0$ ↯

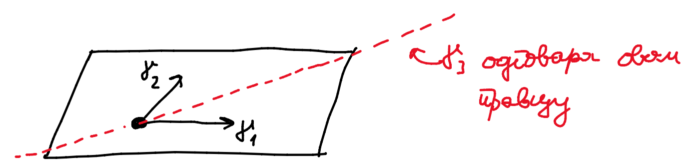
$(A-2E)\gamma_3 = \gamma_2$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

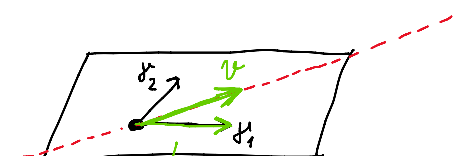
$b+c=0$
$-b-c=1$
$b+c=-1$ ↯

γ_3 не одговара ни γ_1 ни γ_2!

сопствени потпростор:



идеја: променити базу!



нова база за $\ker(A-2E)$

$[\gamma_1, v]$ уместо $[\gamma_1, \gamma_2]$

$v = ?$ $\quad v \in \mathcal{L}in\{\gamma_1, \gamma_2\}$, $\quad v = \alpha\cdot\gamma_1 + \beta\cdot\gamma_2$, $\quad \alpha,\beta = ?$

v – има уопштени γ_3 $\Rightarrow$ $\exists$ решење система $(A-2E)\gamma_3 = v$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta\cdot\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ -\beta \end{bmatrix}$$

$b+c=\alpha$
$-b-c=\beta$

$\alpha = -\beta$

$$b+c=\alpha$$
$$-b-c=\beta$$
$$b+c=-\beta$$
$$\alpha=-\beta$$

$b+c=\alpha$, напр. $\alpha=1, \beta=-1$

$b=1, c=0$, $a=0$

$$\gamma_3=\begin{bmatrix}0\\1\\0\end{bmatrix}, \quad v=\gamma_1-\gamma_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$J=\begin{bmatrix}2&1&\\&2&\\&&2\end{bmatrix} \rightsquigarrow T=\begin{bmatrix}v & \gamma_3 & \gamma_1\end{bmatrix}=\begin{bmatrix}1&0&1\\-1&1&0\\1&0&0\end{bmatrix} \Rightarrow \text{ОР}: y(x)=Te^{xJ}\cdot c, \; c\in\mathbb{R}^3$$

③ $y'=Ay$

$$A=\begin{bmatrix}2&0&1&0\\0&1&0&1\\0&0&2&1\\0&-1&0&1\end{bmatrix} \qquad \lambda_1=\lambda_2=2, \quad \lambda_{3/4}=1\pm i$$

$\lambda_1=\lambda_2=2$:

$k=2$

$$(A-2E)\gamma_1=0 \rightsquigarrow \gamma_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$\Rightarrow m=1$ Ж. блок $\rightsquigarrow$ блок $\begin{bmatrix}2&1\\&2\end{bmatrix}$ у J

$$(A-2E)\gamma_2=\gamma_1 \;\therefore\; \gamma_2=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$\lambda_{3/4}=1\pm i$:

$1+i=\alpha+i\beta \Rightarrow \alpha=\beta=1$

$\rightsquigarrow$ блок $\begin{bmatrix}1&1\\-1&1\end{bmatrix}$ у J

$$(A-(1+i)E)\gamma_3=0$$

$$\gamma_3=\begin{bmatrix}1\\-2\\-1+i\\-2i\end{bmatrix}=\begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}+i\cdot\begin{bmatrix}0\\0\\1\\-2\end{bmatrix}$$

$$J=\begin{bmatrix}2&1&&\\&2&&\\&&1&1\\&&-1&1\end{bmatrix} \rightsquigarrow e^{xJ}=\begin{bmatrix}e^{x\begin{bmatrix}2&1\\&2\end{bmatrix}}&\\&e^{x\begin{bmatrix}1&1\\-1&1\end{bmatrix}}\end{bmatrix}=\begin{bmatrix}e^{2x}&xe^{2x}&&\\0&e^{2x}&&\\&&e^x\cos x&e^x\sin x\\&&-e^x\sin x&e^x\cos x\end{bmatrix}$$

$\hookrightarrow e^x\cdot R_x$

$$T=\begin{bmatrix}\gamma_1 & \gamma_2 & \operatorname{Re}\gamma_3 & \operatorname{Im}\gamma_3\end{bmatrix}=\begin{bmatrix}1&1&1&0\\0&0&-2&0\\0&1&-1&1\\0&0&0&-2\end{bmatrix}$$

ОР: $y(x)=T\cdot e^{xJ}\cdot c, \; c\in\mathbb{R}^4$

4) $y' = Ay$

$$A = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$\lambda_{1/2} = \lambda_{3/4} = \pm i \quad , k=2$

$\lambda_1 = i \longrightarrow \alpha + i\beta = i \Rightarrow \alpha = 0, \beta = 1 \rightsquigarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Могућности за J:

$$J = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{bmatrix}$$

2 Ж. блока

или

једн. матр. реда 2

$$J = \begin{bmatrix} 0 & 1 & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \\ -1 & 0 & \\ & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{bmatrix}$$

1 Ж. блок

$(A - iE)\gamma_1 = 0$

$\therefore \gamma_1 = \begin{bmatrix} 0 \\ -i \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow m = \underbrace{\dim_{\mathbb{C}}}_{\text{комплексна димензија}} Ker(A - \lambda_1 E) = 1 \Rightarrow$ 1 Ж. блок

$$\Rightarrow J = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}$$

Напомена: Ако би се десило да имамо 2 сопствена вектора: γ_1, γ_2

$\Rightarrow m = 2 \Rightarrow J = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix} \rightsquigarrow e^{xJ} = \begin{bmatrix} e^{x\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]} & \\ & e^{x\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]} \end{bmatrix} = \begin{bmatrix} R_x & \\ & R_x \end{bmatrix}$

$T = [Re\gamma_1 \;\; Im\gamma_1 \;\; Re\gamma_2 \;\; Im\gamma_2]$

γ_2 – уопштени за γ_1:

$(A - iE)\gamma_2 = \gamma_1$

$\therefore \gamma_2 = \begin{bmatrix} 1 \\ 0 \\ -1-i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

$$T = [Re\gamma_1 \;\; Im\gamma_1 \;\; Re\gamma_2 \;\; Im\gamma_2] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\dot{J}=\begin{bmatrix} A & E \\ O & A \end{bmatrix}, \quad A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad E=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad O=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\dot{J}=\underbrace{\begin{bmatrix} A & \\ & A \end{bmatrix}}_{=D}+\underbrace{\begin{bmatrix} & E \\ & \end{bmatrix}}_{=N}$$

$$\left.\begin{aligned} DN &= \begin{bmatrix} A & \\ & A \end{bmatrix}\begin{bmatrix} & E \\ & \end{bmatrix}=\begin{bmatrix} O & A \\ O & O \end{bmatrix} \\ ND &= \begin{bmatrix} & E \\ & \end{bmatrix}\begin{bmatrix} A & \\ & A \end{bmatrix}=\begin{bmatrix} O & A \\ O & O \end{bmatrix} \end{aligned}\right\} \overset{2}{\Longrightarrow} e^{x\dot{J}}=e^{xD}\cdot e^{xN}$$

$$e^{xD}=\begin{bmatrix} e^{xA} & \\ & e^{xA} \end{bmatrix}=\begin{bmatrix} R_x & \\ & R_x \end{bmatrix}$$

pega 4

$$e^{xN}=E+xN=\begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} E & xE \\ O & E \end{bmatrix}$$

$$e^{x\dot{J}}=\begin{bmatrix} R_x & O \\ O & R_x \end{bmatrix}\cdot\begin{bmatrix} E & xE \\ O & E \end{bmatrix}=\begin{bmatrix} R_x & xR_x \\ O & R_x \end{bmatrix}$$

$$N^2=\begin{bmatrix} O & E \\ O & O \end{bmatrix}^2=\begin{bmatrix} O & O \\ O & O \end{bmatrix} \Rightarrow N^k=O,\ k\geq 2$$

$$\text{OP}:\ \varphi(x)=Te^{x\dot{J}}\cdot c=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} \cos x & \sin x & x\cos x & x\sin x \\ -\sin x & \cos x & -x\sin x & x\cos x \\ 0 & 0 & \cos x & \sin x \\ 0 & 0 & -\sin x & \cos x \end{bmatrix}\cdot c,\ c\in\mathbb{R}^4$$