Нехомогени ЛСДЈКК

$$y'(x) = \underbrace{A\cdot y(x)}_{\text{хомогени део}} + \underbrace{g(x)}_{\text{нехомогени део}} \qquad (y' = p(x)y + q(x))$$

$$\text{ОР: } y(x) = \underbrace{y_H(x)}_{\substack{\text{ОР хомогене}\\ y'=Ay}} + \underbrace{y_p(x)}_{\substack{\text{ПР нехомогене}\\ y'=Ay+g}}$$

(*) Ако је $g(x) = P_s[x]\cdot e^{\mu x}$, $\mu\in\mathbb{R}$, $s\in\mathbb{N}_0$, $s = st(P_s[x]) = deg(P_s[x])$ $\Big\}$ $\xrightarrow{\text{тражимо у облику}}$ $y_p(x) = Q_{m+s}[x]\cdot e^{\mu x}$

m = вишеструкости броја μ као решења карактеристичне јне A

① а)
$$\begin{aligned} y_1' &= 2y_1 + y_2 + xe^x \\ y_2' &= -y_1 + 2y_2 - e^x \end{aligned}$$

б)
$$\begin{aligned} y_1' &= 2y_1 + y_2 + 2e^x \\ y_2' &= y_1 + 2y_2 - 3e^{4x} \end{aligned}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \qquad y' = Ay + g$$

$$g(x) = \begin{bmatrix} xe^x \\ -e^x \end{bmatrix} = \begin{bmatrix} x \\ -1 \end{bmatrix}\cdot e^x$$

ХОМ: $y' = Ay$

$\det(A-\lambda E)=0 \rightsquigarrow \lambda_{1/2} = 2\pm i.$

$\lambda_1 = 2+i \rightsquigarrow (A-\lambda_1 E)\gamma_1 = 0 \rightsquigarrow \gamma_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$\Psi(x) = e^{\lambda_1 x}\cdot \gamma_1 = e^{2x}\cdot\begin{bmatrix} \cos x + i\sin x \\ i\cos x - \sin x \end{bmatrix} \begin{matrix} \rightsquigarrow Re \\ \rightsquigarrow Im \end{matrix}$$

$$\text{ОРХ: } y_H(x) = c_1\cdot\begin{bmatrix} \cos x \\ -\sin x \end{bmatrix}\cdot e^{2x} + c_2\cdot\begin{bmatrix} \sin x \\ \cos x \end{bmatrix}\cdot e^{2x} \quad, c_1, c_2 \in\mathbb{R}$$

НЕХОМ: $g(x) = \begin{bmatrix} x \\ -1 \end{bmatrix} \cdot e^{x} = P_s[x] \cdot e^{\mu x}$

$\left.\begin{matrix} \mu = 1 \\ s = 1 \end{matrix}\right\} \xrightarrow[\lambda_{1,2} = 2 \pm i]{} \mu$ није сопс. вр. од A

$\Rightarrow m = 0$

$y_p(x) = Q_{s+m}[x] \cdot e^{\mu x} = Q_1[x] \cdot e^{x} = \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \cdot e^{x} \rightsquigarrow y_p' = A y_p + g$

$$\begin{bmatrix} a_1 x + b_1 + a_1 \\ a_2 x + b_2 + a_2 \end{bmatrix} \cdot \cancel{e^{x}} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \cdot \cancel{e^{x}} + \begin{bmatrix} x \\ -1 \end{bmatrix} \cdot \cancel{e^{x}}$$

$\left.\begin{matrix} a_1 x + b_1 + a_1 = 2a_1 x + 2b_1 + a_2 x + b_2 + x \\ a_2 x + b_2 + a_2 = -a_1 x - b_1 + 2a_2 x + 2b_2 - 1 \end{matrix}\right\} \overset{\forall x}{\Longrightarrow}$

$\rightarrow a_1 = 2a_1 + a_2 + 1$

$\rightarrow b_1 + a_1 = 2b_1 + b_2$

$\rightarrow a_2 = -a_1 + 2a_2 \rightsquigarrow a_1 = a_2$

$\rightarrow b_2 + a_2 = -b_1 + 2b_2 - 1$

$a_1 = a_2 = -\frac{1}{2}$

$\left.\begin{matrix} -b_1 - b_2 = \frac{1}{2} \\ -b_2 + b_1 = -\frac{1}{2} \end{matrix}\right\} b_2 = 0,\ b_1 = -\frac{1}{2}$

$y_p(x) = \begin{bmatrix} -\frac{1}{2}x - \frac{1}{2} \\ -\frac{1}{2}x \end{bmatrix} \cdot e^{x}$, ОР: $y(x) = y_H(x) + y_p(x)$

б) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$g(x) = \begin{bmatrix} 2e^{x} \\ -3e^{4x} \end{bmatrix}$

ХОМ: $\lambda_1 = 1,\ \lambda_2 = 3$

$\gamma_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \gamma_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\varphi_1(x) = e^{x} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \varphi_2(x) = e^{3x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

НЕХОМ: $g(x) = \begin{bmatrix} 2e^{x} \\ -3e^{4x} \end{bmatrix} \cancel{=} P_s[x] \cdot e^{\mu x}$

$$g(x) = \begin{bmatrix} 2e^{x} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3e^{4x} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{x} \\ 0 \end{bmatrix} = g_1(x) \rightsquigarrow y_{p_1}(x), \qquad \begin{bmatrix} 0 \\ -3e^{4x} \end{bmatrix} = g_2(x) \rightsquigarrow y_{p_2}(x)$$

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x)$$

$\underline{g_1(x)}$:

$$g_1(x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot e^{x} = P_s[x] \cdot e^{\mu x}$$

$$\left.\begin{matrix} \mu = 1 \\ s = 0 \end{matrix}\right. \xrightarrow[\lambda_1 = 1,\ \lambda_2 = 3]{} m = 1 \Rightarrow y_{p_1}(x) = Q_{0+1}[x] \cdot e^{x} = \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \cdot e^{x}$$

$$y_{p_1}' = A y_{p_1} + g_1$$

$$y_{p_1}(x) = \begin{bmatrix} x - 1 \\ -x \end{bmatrix} \cdot e^{x}$$

шр.

$\underline{g_2(x)}$:

$$g_2(x) = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot e^{4x} = P_s[x] \cdot e^{\mu x}$$

$$\left.\begin{matrix} \mu = 4 \rightsquigarrow m = 0 \\ s = 0 \end{matrix}\right\} \ y_{p_2}(x) = \underset{m+s=0}{Q_0}[x] \cdot e^{4x} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot e^{4x}$$

$$y_{p_2}' = A y_{p_2} + g_2$$

$$4 \begin{bmatrix} a \\ b \end{bmatrix} \cdot \cancel{e^{4x}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \cdot \cancel{e^{4x}} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cancel{e^{4x}}$$

$$\left.\begin{matrix} 4a = 2a + b \\ 4b = a + 2b - 3 \end{matrix}\right\} \ \left.\begin{matrix} a = -1 \\ b = -2 \end{matrix}\right\} \ y_{p_2}(x) = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot e^{4x}$$

$$\text{ОР: } y(x) = C_1 \cdot \begin{bmatrix} e^{x} \\ -e^{x} \end{bmatrix} + C_2 \cdot \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix} + \begin{bmatrix} x - 1 \\ -x \end{bmatrix} \cdot e^{x} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot e^{4x}, \quad C_1, C_2 \in \mathbb{R}$$

$y' = A(x) \cdot y$ → не мора обавезно са КК

ОР: $y(x) = \underline{\Phi(x) \cdot c}$, $c \in \mathbb{R}^n$

$$\Phi(x) = [\varphi_1 \downarrow \dots \varphi_n \downarrow]$$

$$y' = A(x)\,y + g(x)$$

$\Phi(x)$ – фунд. матр. хомогени система $y' = A(x)\cdot y$

ОР: $y(x) = \Phi(x)\cdot\left(c + \int \Phi^{-1}(x)\cdot g(x)\,dx\right) = \underbrace{\Phi(x)\cdot c}_{\text{ОРХ}} + \underbrace{\Phi(x)\cdot\int \Phi^{-1}(x)\cdot g(x)\,dx}_{y_p(x) \text{ пр нехомогене}}$, $c \in \mathbb{R}^n$

② $y_1' = y_2 + tg^2 x + 1$

$y_2' = -y_1 + tg\,x$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} tg^2 x + 1 \\ tg\,x \end{bmatrix} \cancel{\neq} P_s[x]\cdot e^{Mx}$$

$\Phi(x) = ?$

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned}\right\} \quad y_1'' = y_2' = -y_1 \Rightarrow y_1'' + y_1 = 0 \rightsquigarrow y_1(x) = c_1\cos x + c_2 \sin x$$

$$y_2(x) = -y_1'(x) = -c_1 \sin x + c_2 \cos x \qquad c_1, c_2 \in \mathbb{R}$$

$$y_H(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} \cos x \\ -\sin x \end{bmatrix}}_{\varphi_1(x)} + c_2 \underbrace{\begin{bmatrix} \sin x \\ \cos x \end{bmatrix}}_{\varphi_2(x)}$$

$$\Phi(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

$\varphi_1(x)$ $\quad\varphi_2(x)$

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad A^{-1} = \frac{1}{\alpha\delta - \beta\gamma}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$$

$$\det \Phi(x) = W(x) = \cos x \cdot \cos x - \sin x \cdot (-\sin x) = \cos^2 x + \sin^2 x = 1$$

$$\Phi^{-1}(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$g(x) = \begin{bmatrix} tg^2 x + 1 \\ tg\,x \end{bmatrix} = \begin{bmatrix} \frac{\sin^2 x}{\cos^2 x} + 1 \\ tg\,x \end{bmatrix} = \begin{bmatrix} \frac{\sin^2 x + \cos^2 x}{\cos^2 x} \\ tg\,x \end{bmatrix} = \begin{bmatrix} \frac{1}{\cos^2 x} \\ \frac{\sin x}{\cos x} \end{bmatrix}$$

$$\frac{1 - \sin^2 x}{\cos x} = \frac{\cos^2 x}{\cos x} = \cos x$$

$$\int \Phi^{-1}(x)\cdot g(x)\,dx = \int \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\cos^2 x} \\ \frac{\sin x}{\cos x} \end{bmatrix} dx = \int \begin{bmatrix} \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x} \\ \frac{\sin x}{\cos^2 x} + \sin x \end{bmatrix} dx = \int \begin{bmatrix} \cos x \\ \frac{\sin x}{\cos^2 x} + \sin x \end{bmatrix} dx$$

↳ члан по члан

$$\begin{bmatrix} \sin x \\ \end{bmatrix}$$

→ члан по члан

$$\int \cos x\, dx = \sin x$$

$$\int \sin x\, dx = -\cos x$$

$$\int \frac{\sin x}{\cos^2 x} dx = \int \frac{-du}{u^2} = \frac{1}{u} = \frac{1}{\cos x}$$

$u = \cos x$

$du = -\sin x\, dx$

$$= \begin{bmatrix} \sin x \\ \frac{1}{\cos x} - \cos x \end{bmatrix}$$

$$y_p(x) = \Phi(x) \cdot \int \ldots = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} \sin x \\ \frac{1}{\cos x} - \cos x \end{bmatrix} = \begin{bmatrix} \cos x \cdot \sin x + \frac{\sin x}{\cos x} - \sin x \cdot \cos x \\ \underbrace{-\sin^2 x + 1 - \cos^2 x}_{=1} \end{bmatrix} = \begin{bmatrix} \operatorname{tg} x \\ 0 \end{bmatrix}.$$

$$\text{OP}: \quad y(x) = \Phi(x) \cdot c + y_p(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \operatorname{tg} x \\ 0 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

Експоненти матрице

$n \in \mathbb{N}$, $A \in M_n(\mathbb{R})$ $(\mathbb{C})$ $\rightsquigarrow$ $\exp : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$A \in M_n(\mathbb{R}) \rightsquigarrow A^k \in M_n(\mathbb{R})$

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$$\left[e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \; x \in \mathbb{R} \right]$$

Особине: $(\forall A, B \in M_n(\mathbb{R}))$

1. $\frac{d}{dt}(e^{tA}) = A \cdot e^{tA} = e^{tA} \cdot A$
2. $AB = BA \Rightarrow e^{A+B} = e^A \cdot e^B$
3. $AB = BA \Rightarrow Ae^B = e^B A$ (спец: $B = A \Rightarrow Ae^A = e^A A$)
4. $\det(e^A) = e^{\operatorname{tr} A}$
5. $\lim_{n\to\infty} \left(E + \frac{A}{n}\right)^n = e^A$ $\quad$ $\left(1 + \frac{x}{n}\right)^n \to e^x$

③ Испитати да ли постоји $A\in M_2(\mathbb{R})$ тд:

а) $e^A=\begin{bmatrix}1&0\\0&-4\end{bmatrix}$ б) $e^A=\begin{bmatrix}-1&0\\0&-4\end{bmatrix}$ ($e^a=b$, $b>0$ - има решења)

а) ④ $\Rightarrow$
$$\left.\begin{array}{l}\det(e^A)=e^{trA}\\ \quad \parallel \\ \det\left(\begin{bmatrix}1&0\\0&-4\end{bmatrix}\right)=1\cdot(-4)=-4\end{array}\right\}\ e^{trA}=-4\ \text{↯}\ \nexists A$$

б)
$$\left.\begin{array}{l}\det(e^A)=e^{trA}\\ \quad \parallel \\ \det\left(\begin{bmatrix}-1&0\\0&-4\end{bmatrix}\right)=4\end{array}\right\}\ trA=\ln 4$$

③ $\underset{B=A}{\Rightarrow}$ $Ae^A=e^AA$, $A=\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}$

$$\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}\begin{bmatrix}-1&0\\0&-4\end{bmatrix}=\begin{bmatrix}-1&0\\0&-4\end{bmatrix}\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}$$

$$\cancel{-\alpha=-\alpha}$$
$$\left.\begin{array}{l}-4\beta=-\beta\\ -\gamma=-4\gamma\end{array}\right\}\ \beta=\gamma=0\Rightarrow A=\begin{bmatrix}\alpha&0\\0&\delta\end{bmatrix}$$
$$\cancel{-4\delta=-4\delta}$$

$$A^2=\begin{bmatrix}\alpha^2&0\\0&\delta^2\end{bmatrix},\ A^3=\begin{bmatrix}\alpha^3&0\\0&\delta^3\end{bmatrix},\ \dots,\ A^k=\begin{bmatrix}\alpha^k&0\\0&\delta^k\end{bmatrix}$$

$$e^A=\sum_{k=0}^{\infty}\frac{A^k}{k!}=\sum_{k=0}^{\infty}\frac{\begin{bmatrix}\alpha^k&0\\0&\delta^k\end{bmatrix}}{k!}=\begin{bmatrix}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}&0\\0&\sum_{k=0}^{\infty}\frac{\delta^k}{k!}\end{bmatrix}=\begin{bmatrix}e^\alpha&0\\0&e^\delta\end{bmatrix}.$$

$$\begin{bmatrix}e^\alpha&0\\0&e^\delta\end{bmatrix}=\begin{bmatrix}-1&0\\0&-4\end{bmatrix}\Rightarrow\begin{array}{l}e^\alpha=-1\\e^\delta=-4\end{array}\ \text{↯}\Rightarrow\nexists A$$

⊛ $A=diag[\lambda_1\dots\lambda_n]=\begin{bmatrix}\lambda_1&&0\\&\ddots&\\0&&\lambda_n\end{bmatrix}\Rightarrow e^A=diag[e^{\lambda_1}\dots e^{\lambda_n}]$

④ Нека су $A,B\in M_n(\mathbb{R})$ и нека $\exists B^{-1}$. Доказати $e^{B^{-1}AB}=B^{-1}e^AB$.

$e^{B^{-1}AB}\ \sum^{\infty}(B^{-1}AB)^k\ \sum^{\infty}(B^{-1})A^kB$ …

$$e^{B^{-1}AB} = \sum_{k=0}^{\infty} \frac{(B^{-1}AB)^k}{k!} = \sum_{k=0}^{\infty} \frac{B^{-1}A^kB}{k!} = B^{-1} \cdot \left(\sum_{k=0}^{\infty} \frac{A^k}{k!} \right) \cdot B = B^{-1} \cdot e^{A} \cdot B$$

$$(B^{-1}AB)^k = (B^{-1}AB) \cdot (B^{-1}AB) \cdot (B^{-1}AB) \cdot \ldots \cdot (B^{-1}AB) =$$

$$= B^{-1}A\overbrace{(BB^{-1})}^{E}A\overbrace{(BB^{-1})}^{E}A(B \cdot \ldots \cdot \ldots B^{-1})AB =$$

$$= B^{-1}AAA \ldots AB =$$

$$= B^{-1}A^kB$$

(*) множење матрица није комутативно, али јесте асоцијативно