(1) Свести систем ДЈ на систем ДЈ у нормалном облику.

$$y' = f(x, y)$$

$$\begin{cases} y_1' = f_1(x, y_1, \dots, y_n) \\ y_2' = f_2(x, y_1, \dots, y_n) \\ \vdots \\ y_n' = f_n(x, y_1, \dots, y_n) \end{cases}$$

скуп неколико ДЈ

$y_1, y_2, \dots, y_n$ – непознате фје

$x$ – независно променљива

→ систем дј у коме фигуришу

$x, y_j, y_j', y_j'', \dots$ бј

$$\begin{cases} y''' = xy'' + y + z' + 2x - 1 \\ z'' = y' \cos x + \sin(y \cdot z') \end{cases}$$

3. реда ($y'''$), 2. реда ($z''$)

5 јна 1. реда

За непознате: $y, z$ и још: $y', y'', z'$ (означене $y_1$, $y_2$, $z_1$)

ово даје 2 јне:

$$\begin{cases} y_2' = x \cdot y_2 + y + z_1 + 2x - 1 \\ z_1' = y_1 \cdot \cos x + \sin(y \cdot z_1) \\ y' = y_1 \\ z' = z_1 \\ y_1' = y_2 \end{cases}$$

систем реда 5

$y' = y_1$

$z' = z_1$

$y_1' = (y')' = y'' = y_2$

(2) Методом елиминације решити системе ДЈ:

а) $y' = py - qz$

$z' = qy + pz$

$p, q \in \mathbb{R} \setminus \{0\}$

Изразити из једне и убацити у другу

б) $y_1' = y_2$

$y_2' = y_1$

$y_3' = y_1 + y_2 + y_3$

а) $y' = py - qz \Rightarrow z = \frac{1}{q}(py - y') \Rightarrow z' = \frac{1}{q}(py' - y'')$ ($q \neq 0$)

$z' = qy + pz$

$$\frac{1}{q}(py' - y'') = qy + p \cdot \frac{1}{q}(py - y') \quad / \cdot q$$

$$py' - y'' = q^2 y + p^2 y - py'$$

$$y'' - 2py' + (p^2+q^2)y = 0 \rightarrow \text{ЛХДЈКК 2. реда}$$

$$\lambda^2 - 2p\lambda + (p^2+q^2) = 0$$

$$D = (-2p)^2 - 4\cdot(p^2+q^2) = 4p^2 - 4p^2 - 4q^2 = -4q^2 < 0$$

$$\lambda_{1/2} = \frac{2p \pm i\cdot 2q}{2} = p \pm iq$$

ОР: $y(x) = c_1 e^{px}\cos qx + c_2 e^{px}\sin qx, \quad c_1, c_2 \in \mathbb{R}$

I) $z = \frac{1}{q}(py - y')$ ✓ директно добијамо $z$

II) $z' = qy + pz$ ✗ решавањем ове дј. добијамо још 1 константу → укупно 3
реда 2 → 2 константе
(морала би провера)

$$z = \frac{1}{q}(py - y') = \text{---} = c_1 e^{px}\sin qx - c_2 e^{px}\cos qx$$

$$y' = c_1 (e^{px})'\cos qx + c_1 e^{px}\cdot(\cos qx)' + c_2 (e^{px})'\sin qx + c_2 e^{px}\cdot(\sin qx)' =$$

$$= c_1 p e^{px}\cos qx - c_1 q e^{px}\sin qx + c_2 p e^{px}\sin qx + c_2 q e^{px}\cos qx$$

б)

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 \end{aligned}\right\} \rightarrow \text{2 јне, 2 непознате}$$

$$y_3' = y_1 + y_2 + y_3$$

$y_1' = y_2 \Rightarrow y_2' = y_1''$

$y_2' = y_1 \Rightarrow y_1'' = y_1 \xrightarrow{\lambda^2 - 1 = 0} \begin{cases} y_1(x) = c_1 e^{x} + c_2 e^{-x}, & c_1, c_2 \in \mathbb{R} \\ y_2(x) = y_1'(x) = c_1 e^{x} - c_2 e^{-x} \end{cases}$

$$y_3' - y_3 = y_1 + y_2 = 2c_1 e^{x} \quad \text{(лин. јна.)}$$

$p(x) = -1$

$q(x) = 2c_1 e^{x}$

$\int p(x)\,dx = -x$

$\int q(x)\cdot e^{\int p(x)dx}\,dx = \int 2c_1 e^{x}\cdot e^{-x}\,dx = 2c_1 \int dx = 2c_1 x$

$$y_3(x) = e^{-\int p(x)dx}\cdot\left(c_3 + \int q(x)\cdot e^{\int p(x)dx}\,dx\right) = e^{x}(c_3 + 2c_1 x)$$

$c_3 \in \mathbb{R}$

③ Помоћу смене независне променљиве решити систем ДЈ:

а) $2\sqrt{x}\cdot y_1' = 2y_1 - y_2$

$2\sqrt{x}\cdot y_2' = y_1 + 2y_2$

б) $x\, y_1' = y_1 + y_2$

$x\, y_2' = y_2$

$x\, y_3' = -y_3$

напомена за б:

$xy_2' = y_2 \Rightarrow \frac{dy_2}{y_2} = \frac{dx}{x}$ (РП) $\rightsquigarrow y_2(x)$

$xy_3' = -y_3 \Rightarrow \frac{dy_3}{y_3} = -\frac{dx}{x}$ (РП) $\rightsquigarrow y_3(x)$

$y_1' = \frac{1}{x}\cdot y_1 + \frac{y_2}{x}$ (лин.) $\rightsquigarrow y_1(x)$

може је и директно

$x \rightsquigarrow t: \; y_1(x) \rightsquigarrow y_1(t)$

$y_1' = \frac{dy_1}{dx} \rightsquigarrow y_{1t}' = \frac{dy_1}{dt}$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad Y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}, \quad Y: \mathbb{R} \to \mathbb{R}^n$$

$$Y' = F(x, Y)$$

$$\parallel$$

$$(f_1, f_2, \dots, f_n)$$

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$f(x)\cdot Y' = F(x, Y)$

$f(x) = 2\sqrt{x}, x, \dots$

$t(x), \; x(t)$

$\frac{dy_1}{dx} = \frac{dy_1}{dt}\cdot\frac{dt}{dx}$

хоћемо

$f(x)\cdot\frac{dy_1}{dx} = \frac{dy_1}{dt}$

$\parallel$

$f(x)\cdot\frac{dy_1}{dt}\cdot\frac{dt}{dx}$

треба: $f(x)\cdot\frac{dt}{dx} = 1$

$\frac{dt}{dx} = \frac{1}{f(x)} \;/\int$

$$\boxed{t = \int \frac{dx}{f(x)}}$$

а) $f(x) = 2\sqrt{x}, \quad x \geq 0$

$t = \int \frac{dx}{2\sqrt{x}} = \sqrt{x}, \quad x = t^2$

$\frac{dy_1}{dx} = \frac{dy_1}{dt}\cdot\frac{dt}{dx} = \frac{dy_1}{dt}\cdot\frac{1}{2\sqrt{x}} \Rightarrow \frac{dy_1}{dt} = y_{1t}' = 2\sqrt{x}\cdot\frac{dy_1}{dx} = 2\sqrt{x}\cdot y_1'$

систем: $y_{1t}' = 2y_1 - y_2 \Rightarrow y_2 = 2y_1 - y_1'$

система: $y'_{1t} = 2y_1 - y_2 \Rightarrow y_2 = 2y_1 - y'_{1t}$

$y'_{2t} = y_1 + 2y_2$

$\Downarrow$

$2y'_{1t} - y''_{1t} = y_1 + 4y_1 - 2y'_{1t}$

$y''_{1t} - 4y'_{1t} + 5y_1 = 0$

$\vdots$

$\lambda_{1/2} = 2 \pm i$

$y_1(t) = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t, \quad c_1, c_2 \in \mathbb{R}$

$y_2(t) = 2y_1 - y'_{1t} = \ldots$

ОР: $\left.\begin{array}{l} y_1(x) = c_1 e^{2\sqrt{x}}\cos\sqrt{x} + c_2 e^{2\sqrt{x}}\sin\sqrt{x} \\ y_2(x) = -c_2 e^{2\sqrt{x}}\cos\sqrt{x} + c_1 e^{2\sqrt{x}}\sin\sqrt{x} \end{array}\right\} \quad c_1, c_2 \in \mathbb{R}, \quad x > 0$

$x = 0$: $2y_1(0) - y_2(0) = y_1(0) + 2y_2(0) = 0 \Rightarrow y_1(0) = y_2(0) = 0$

→ не могу се додефинисати два решења у $x = 0$

б) $t = \int \frac{dx}{f(x)} = \int \frac{dx}{x} = \ln|x|$

$f(x) = x$

1° $x > 0$, $t = \ln x$, $x = e^t$

$y'_{1t} = \frac{dy_1}{dt} = \frac{dy_1}{dx} \cdot \frac{dx}{dt} = y_1' \cdot \underbrace{e^t}_{x} = x \cdot y_1'$

2° $x < 0$, $t = \ln(-x)$, $x = -e^t$

$\vdots$

домаћи

система: $y'_{1t} = y_1 + y_2$

$y'_{2t} = y_2 \Rightarrow y_2(t) = c_1 e^t$ (= $x$)

$y'_{3t} = -y_3 \Rightarrow y_3(t) = c_2 e^{-t}$ (= $\frac{1}{x}$)

→ $y'_{1t} - y_1 = c_1 e^t$ (лин)

$\vdots$

$y_1(t) = e^t (c_3 + c_1 t)$ (where $e^t$ = $x$, $t$ = $\ln x$)

$$\text{OP}: \ Y(x) = \begin{bmatrix} x(c_3 + c_1 \ln x) \\ c_1 x \\ \frac{c_2}{x} \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \in \mathbb{R}^3$$

$x=0$? ... $y_1(0) = y_2(0) = y_3(0) = 0$

④ Методом елиминације решити системе ДЈ:

а) $y' = 1 - \frac{1}{z}$, $\quad z' = \frac{1}{y-x}$

б) $2zy' = y^2 - z^2 + 1$, $\quad z' = y + z$

а) $z \neq 0$

$y \neq x$

$$z' = \frac{1}{y-x} \overset{z' \neq 0}{\Rightarrow} \frac{1}{z'} = y - x \Rightarrow y = x + \frac{1}{z'} \ /'$$

$$\Downarrow$$

$$y' = 1 + \left(\frac{1}{z'}\right)' = 1 - \frac{1}{z'^2} \cdot z''$$

$$y' = 1 - \frac{1}{z} \Rightarrow 1 - \frac{z''}{z'^2} = 1 - \frac{1}{z} \Rightarrow \frac{z''}{z'^2} = \frac{1}{z} \Rightarrow z'' \cdot z = z'^2 \ /: z^2$$

нелинеарна 2. реда

нема $x$

$$\left(\frac{z'}{z}\right)' = \frac{(z')' \cdot z - z' \cdot (z)'}{z^2} = \frac{z'' z - z'^2}{z^2}$$

$$\frac{z'' \cdot z - z'^2}{z^2} = 0$$

$$\left(\frac{z'}{z}\right)' = 0 \ / \int$$

$$\frac{z'}{z} = c_1, \quad c_1 \in \mathbb{R}$$

$$z' = c_1 z$$

$$\vdots$$

$$z = c_2 \cdot e^{c_1 x}, \quad c_2 \in \mathbb{R} \setminus \{0\}$$

$$z' = c_1 c_2 e^{c_1 z} \neq 0$$

$$y = x + \frac{1}{z'} = x + \frac{1}{c_1 c_2} e^{-c_1 z}$$

$$\text{OP}: \quad \left.\begin{aligned} y &= x + \frac{e^{-c_1 z}}{c_1 c_2} \\ z &= c_2 e^{c_1 x} \end{aligned}\right\} \quad c_1, c_2 \in \mathbb{R}\setminus\{0\}$$

б) $2zy' = y^2 - z^2 + 1$

$z' = y + z \;\to\; y = z' - z$

$y' = z'' - z'$

$$2z \cdot (z'' - z') = (z' - z)^2 - z^2 + 1$$

$$2zz'' - 2zz' = z'^2 - 2z'z + z^2 - z^2 + 1$$

$$2zz'' = z'^2 + 1 \quad \leftarrow \text{нелинеарна 2. реда, нема } x$$

СМЕНА: $u(z) = z'$

$u'(z) = \frac{d}{dz}(z') = \frac{dz'}{dx} \cdot \frac{dx}{dz} = z'' \cdot \frac{1}{z'}$

$\Rightarrow z'' = u' \cdot z' = uu'$

$$2z \cdot uu' = u^2 + 1$$

$$\frac{2uu'}{u^2+1} = \frac{1}{z} \quad (\text{РП})$$

$$\int \frac{2u\,du}{u^2+1} = \int \frac{dz}{z}$$

$$\ln(u^2+1) = \ln|z| + c_1 \quad , c_1 \in \mathbb{R}$$

$$u^2 + 1 = |z| \cdot e^{c_1} = c_2 \cdot z \quad , \quad c_2 \in \mathbb{R}\setminus\{0\}; \qquad c_2 = 0 \times$$

$$z'^2 + 1 = c_2 z$$

$$z'^2 = c_2 z - 1 \quad (c_2 z - 1 \geq 0)$$

$$z' = \pm\sqrt{c_2 z - 1}$$

1° $z' = \sqrt{c_2 z - 1}$ (РП)

$$\frac{dz}{\sqrt{c_2 z - 1}} = dx \;\Big/ \int$$

$$\frac{1}{c_2} 2\sqrt{c_2 z - 1} = x + c_3 \quad , c_3 \in \mathbb{R}$$

$\Downarrow$

⋮

2° $z' = -\sqrt{c_2 z - 1}$

⋮

$$\frac{1}{c_2} 2\sqrt{c_2 z - 1} = x + c_3, \quad c_3 \in \mathbb{R}$$

$$\Downarrow$$

$$z = \frac{1}{c_2}\left(1 + \frac{c_2^2}{4}(x+c_3)^2\right) \rightsquigarrow y = z' - z = \frac{c_2}{2}(x+c_3) - \frac{1}{c_2} - \frac{c_2}{4}(x+c_3)^2$$

$$z' = \frac{c_2}{2}(x+c_3)$$