(1)
$$\left.\begin{aligned} x' &= 4y-3z \\ y' &= 4x-2z \\ z' &= 2y-3x \end{aligned}\right\} \text{2 прва интеграла}$$

домаћи: $X'=AX$ решити експлицитно и упоредити

идеја: метод линеарних комбинација

$$\alpha x' + \beta y' = z' \quad (\alpha x' + \beta y' + \gamma z' = 0), \ \alpha, \beta \in \mathbb{R}$$

$\alpha, \beta = ?$

$\alpha x' + \beta y' = z'$

$$\alpha(4y-3z) + \beta(4x-2z) = 2y-3x$$
$$\parallel$$
$$x(4\beta) + y(4\alpha) + z(-3\alpha-2\beta)$$

$$4\beta = -3$$
$$4\alpha = 2$$
$$-3\alpha - 2\beta = 0$$

$$\alpha = \frac{1}{2}, \ \beta = -\frac{3}{4} \quad \Rightarrow \quad \frac{1}{2}x' - \frac{3}{4}y' = z' \ \Big/ \int dt$$

$$\frac{x}{2} - \frac{3}{4}y = z + c_1$$

$$\underbrace{\frac{x}{2} - \frac{3}{4}y - z}_{\psi_1(x,y,z)} = c_1$$

$\alpha, \beta = ?$

$$\alpha x x' + \beta y y' = z z' \Rightarrow \alpha x(4y-3z) + \beta y(4x-2z) = z(2y-3x) = 2yz - 3xz$$
$$\parallel$$
$$xy(4\alpha+4\beta) + xz(-3\alpha) + yz(-2\beta)$$

$$\int x' dt = x + c$$
$$\int x x' dt = \frac{x^2}{2} + c$$

$$4\alpha + 4\beta = 0$$
$$-3\alpha = -3$$
$$-2\beta = 2$$

$$\alpha = 1, \ \beta = -1 \quad \Rightarrow \quad xx' - yy' = zz' \ \Big/ dt$$

$$\underbrace{\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2}}_{\psi_2(x,y,z)} = c_2$$

$$\nabla \psi_1 = \left(\frac{1}{2}, -\frac{3}{4}, -1\right)$$

$$\nabla\psi_2 = (x, -y, -z)$$

$$\nabla\psi_1 \times \nabla\psi_2 = \begin{vmatrix} \frac{1}{2} & -\frac{3}{4} & -1 \\ x & -y & -z \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} = \left(\frac{3}{4}z - y\right)\vec{i} - \left(-\frac{z}{2} + x\right)\vec{j} + \left(-\frac{y}{2} + \frac{3}{4}x\right)\vec{k} \neq 0$$

осим ~~ако~~ $3z = 4y$

$z = 2x$

$y = \frac{3}{2}x$

једини проблем $(x, \frac{3}{2}x, 2x), x \in \mathbb{R}$

$$C_1 = \frac{x}{2} - \frac{3}{4}\cdot\frac{3}{2}x - 2x = \frac{4x - 9x - 16x}{8} = -\frac{21}{8}x$$

$$C_2 = \frac{x^2}{2} - \frac{1}{2}\cdot\frac{9}{4}x^2 - \frac{1}{2}\cdot 4x^2 = \frac{x^2(4 - 9 - 16)}{8} = -\frac{21}{8}x^2$$

$$C_1^2 = -\frac{21}{8}\cdot C_2 \rightarrow \text{ово не сме}$$

OP: $\psi_1 = C_1$ $\quad C_1^2 \neq -\frac{21}{8}C_2$

$\psi_2 = C_2$

(2) $x' = x(y^2 + z^2)$

$y' = -y(x^2 + z^2)$

$z' = z(y^2 - x^2)$

$$\alpha x x' + \beta y y' = z z'$$

$$\alpha x^2(y^2 + z^2) - \beta y^2(x^2 + z^2) = z^2(y^2 - x^2)$$

$\alpha = -1$

$\beta = -1 \quad \Rightarrow \quad -xx' - yy' = zz'$

$\vdots$

$$\psi_1(x, y, z) = x^2 + y^2 + z^2$$

$$\frac{\alpha}{x}x' + \frac{\beta}{y}y' = \frac{1}{z}z'$$

$$\alpha(y^2 + z^2) - \beta(x^2 + z^2) = y^2 - x^2 \qquad \int \frac{x'}{x}dt = \ln|x| + c$$

$\alpha = 1$

$\beta = 1 \quad \Rightarrow \quad \frac{x'}{x} + \frac{y'}{y} = \frac{z'}{z} \ / \int dt$

$$\ln|x| + \ln|y| = \ln|z| + c_1$$

$$\ln\left|\frac{xy}{z}\right| = C_1 \quad \dots \quad \underbrace{\frac{xy}{z}}_{\psi_2(x,y,z)} = C_2$$

проверити независност...

домаћи:
$$x' = \frac{x(t^2+y^2)}{t(y^2-x^2)}$$
$$y' = \frac{y(t^2+x^2)}{t(x^2-y^2)}$$
урадити слично као прошли зад.

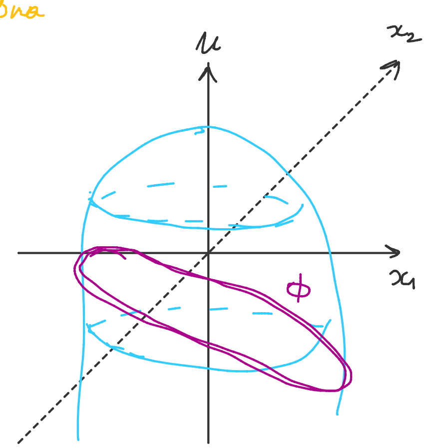
Парцијалне ДЈ 1. реда (ПДЈ)

$$u(x_1, \dots, x_n) = ? \rightsquigarrow F\left(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0$$
1. реда

квазилинеарна (КЛ):
$$\sum_{j=1}^{n} a_j(x_1, \dots, x_n, u)\frac{\partial u}{\partial x_j} = c(x_1, \dots, x_n, u)$$
сметa да буде линеарна

хомогено линеарна (ХЛ):
$$\sum_{j=1}^{n} a_j(x_1, \dots, x_n)\frac{\partial u}{\partial x_j} = 0 \quad (c=0)$$

Кошијев проблем: наћи решење које садржи задату фигуру ϕ.
$$\phi \subseteq \Gamma(u)$$

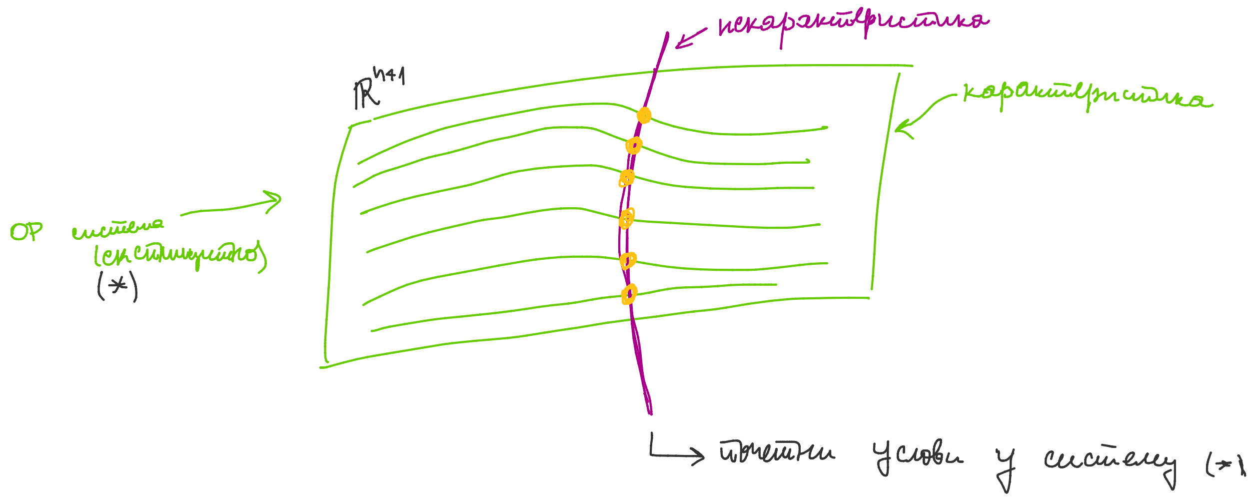


(КЛ) $\Rightarrow$
$$x_j'(t) = a_j(x_1, \dots, x_n, u)$$
$$u'(t) = c(x_1, \dots, x_n, u)$$

систем карактеристика (*)

(ХЛ) $\Rightarrow$
$$x_j'(t) = a_j(x_1, \dots, x_n)$$

Метода карактеристика (за решавање Кошијевог проблема)



- погледати предавања (2 урађена задатка)
- по потреби прошлу годину (још 2 задатка)

Метода првих интеграла

идеја: у систему (*) наћи прве интеграле који су независни

③ $x(y^2-z^2)\cdot\frac{\partial u}{\partial x} - y(x^2+z^2)\frac{\partial u}{\partial y} + z(x^2+y^2)\frac{\partial u}{\partial z} = 0$ $\qquad u\big|_{x=1} = (y+z)^2$

нема u $\qquad$ површ

(ХЛ) $\quad u(x,y,z)$

$$\left.\begin{aligned} x' &= x(y^2-z^2) \\ y' &= -y(x^2+z^2) \\ z' &= z(x^2+y^2) \end{aligned}\right\} \text{ 2 прва инт.}$$

слично је као зад ② за вежбу $\qquad \psi_1 = x^2+y^2+z^2$, $\quad \psi_2 = \frac{yz}{x}$

$$\boxed{\text{ОР}: \; u = \varphi(\psi_1, \psi_2)\,, \; \varphi \in C^1(\mathbb{R}^2)}$$

ОР за хл јну

Наћи φ тд. је задовољен Кошијев услов $u\big|_{x=1} = (y+z)^2$

Провери

$$u(1,y,z) = (y+z)^2$$
$$\shortparallel$$
$$\varphi(\psi_1(1,y,z),\ \psi_2(1,y,z)) = (y+z)^2$$
$$\shortparallel$$
$$\varphi(1+y^2+z^2,\ yz) = (y+z)^2$$

узмемо $\varphi(X,Y) = X-1+2Y$

Кош. реш: $u = \varphi(\psi_1, \psi_2) = \psi_1 - 1 + 2\psi_2 = x^2+y^2+z^2-1+2\frac{yz}{x}$

(4) $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} + z^2 = 0$, $z=1$, $xy=x+y$

$$x' = x^2$$
$$y' = y^2$$
$$z' = -z^2$$
(КЛ)

$$\frac{x'}{y'} = \frac{x^2}{y^2} \Rightarrow \frac{x'}{x^2} = \frac{y'}{y^2} \Big/ \int dt$$

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\psi_1(x,y,z) = \frac{1}{x} - \frac{1}{y}$$

$$\frac{x'}{z'} = \frac{x^2}{-z^2} \Rightarrow \frac{x'}{x^2} = -\frac{z'}{z^2} \Big/ \int dt$$

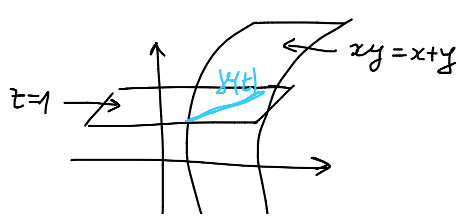
$$\psi_2(x,y,z) = \frac{1}{x} + \frac{1}{z}$$

независност: $\nabla\psi_1 \times \nabla\psi_2 = \ldots = \left(-\frac{1}{y^2z^2},\ -\frac{1}{x^2z^2},\ \frac{1}{x^2y^2}\right) \neq \vec{0} \Rightarrow \psi_1$ и ψ_2 нез.

$$\boxed{OP:\ \varphi(\psi_1, \psi_2) = 0,\ \varphi \in C^1(\mathbb{R}^2)}$$

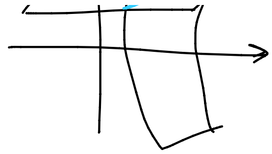
OP за КЛ јнњ (имплицитно задато)

КР: Наћи φ т.д. $\varphi(\psi_1,\psi_2)=0$ и та фигура садржи криву $\left.\begin{matrix} z=1 \\ xy=x+y \end{matrix}\right\}$



$$\gamma(t) = \left(t,\ \frac{t}{t-1},\ 1\right)$$

$$xy = x+y$$
$$y(x-1) = x$$
$$y = \frac{x}{x-1}$$



$\gamma(t) = \left(t, \frac{t}{t-1}, 1\right)$ $\quad x-1$

$\psi_1(\gamma(t)) = \frac{1}{t} - \frac{1}{\frac{t}{t-1}} = \frac{1-t+1}{t} = \frac{2-t}{t}$

$\psi_2(\gamma(t)) = \frac{1}{t} + 1 = \frac{1+t}{t}$

φ? $\quad \varphi(\psi_1|_\gamma, \psi_2|_\gamma) = 0$

$\Rightarrow \varphi\left(\frac{2-t}{t}, \frac{1+t}{t}\right) = 0$

$\varphi(X, Y) = X - 2Y + 3$

КР: $\quad \psi_1 - 2\psi_2 + 3 = 0$

$\frac{1}{x} - \frac{1}{y} - \frac{2}{x} - \frac{2}{z} + 3 = 0$

$-\frac{1}{x} - \frac{1}{y} - \frac{2}{z} = -3$

$\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = 3$ ← иниц. зад.

чак може и експл. овде:

$z = \dots = \frac{2}{3 - \left(\frac{1}{x} + \frac{1}{y}\right)}$

(5) $\quad x(x^2+3y^2)\frac{\partial z}{\partial x} + y(3x^2+y^2)\frac{\partial z}{\partial y} = 2z(x^2+y^2)$ $\quad$ (КЛ)

$xy = z$
$x^2 - y^2 = z^2$

$x' = x(x^2+3y^2)$

$y' = y(3x^2+y^2)$ $\quad \rightsquigarrow \psi_1$ и ψ_2 ?

$z' = 2z(x^2+y^2)$

$\frac{\alpha x'}{x} + \frac{\beta y'}{y} = \frac{z'}{z}$

$\alpha(x^2+3y^2) + \beta(3x^2+y^2) = 2(x^2+y^2)$

$\left.\begin{aligned} \alpha + 3\beta &= 2 \\ 3\alpha + \beta &= 2 \end{aligned}\right\} \quad \alpha = \beta = \frac{1}{2} \Rightarrow \frac{x'}{2x} + \frac{y'}{2y} = \frac{z'}{z} \Big/ \int dt$

$\frac{1}{2}\ln|x| + \frac{1}{2}\ln|y| = \ln|z| + c_1$

$\vdots$

$\frac{xy}{z^2} = \tilde{c}_1 = \psi_1(x, y, z)$

$$xx' - yy' = x^2(x^2+3y^2) - y^2(3x^2+y^2) = x^4 - y^4 = (x^2-y^2)(x^2+y^2) = (x^2-y^2)\cdot\frac{z'}{2z}$$

$$2\cdot\frac{xx'-yy'}{x^2-y^2} = \frac{z'}{z} \Big/ \int dt \Rightarrow \ln|x^2-y^2| = \ln|z| + c_2$$

$$\psi_2(x,y,z) = \frac{x^2-y^2}{z}$$

$$\left(\ln|x^2-y^2|\right)' = \frac{1}{|x^2-y^2|}\cdot \operatorname{sgn}(x^2-y^2)\cdot(x^2-y^2)' = \frac{2xx'-2yy'}{x^2-y^2}$$

Независности: домаћи

$$\text{ОР}: \varphi(\psi_1,\psi_2) = 0, \ \varphi\in C^1(\mathbb{R}^2)$$

КР: $\left.\begin{array}{l} xy = z \\ x^2-y^2 = z^2 \end{array}\right\}$ у пресеку је крива c

$$\psi_1|_c = \left(\frac{xy}{z^2}\right)\Big|_c \underset{xy=z}{=} \frac{z}{z^2} = \frac{1}{z}$$

$$\psi_2|_c = \left(\frac{x^2-y^2}{z}\right)\Big|_c \underset{x^2-y^2=z^2}{=} \frac{z^2}{z} = z$$

$\psi_1|_c$ и $\psi_2|_c$ се задају преко 1 параметра

$$\varphi = ?, \quad \varphi(\psi_1|_c, \psi_2|_c) = 0 \Rightarrow \varphi\left(\frac{1}{z}, z\right) = 0$$

$$\varphi(X,Y) = X\cdot Y - 1$$

Кошијево реш: $\varphi(\psi_1,\psi_2) = 0$

$$\psi_1\cdot\psi_2 - 1 = 0$$

$$\frac{xy}{z^2}\cdot\frac{x^2-y^2}{z} = 1 \Rightarrow z^3 = xy(x^2-y^2) \Rightarrow z = \sqrt[3]{xy(x^2-y^2)}$$