① $X' = AX$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$

$k = 4$

$$(A-2E)\gamma = 0 \rightsquigarrow \gamma \in \mathcal{L}in \left\{ \overset{\gamma_1}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{\gamma_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \right\} \Rightarrow m = 2 \Rightarrow 2 \text{ Ж. блока}$$

$$\begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix} \vee \underbrace{\begin{bmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{bmatrix}}$$

$\varphi(\lambda) = (\lambda-2)^4$

$\mu(\lambda) = (\lambda-2)^\ell, \; \ell \le 4$

$\mu(A) = 0, \quad A - 2E \ne 0$

$(A-2E)^2 \ne 0$

$(A-2E)^3 = 0 \Rightarrow \deg\mu = 3 \Rightarrow$ ред највећег блока је 3 $\Rightarrow J = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & \\ & \boxed{2} \end{bmatrix}$

уопштени за γ_2:

$(A-2E)\gamma = \gamma_2 \quad \Rightarrow$ нема реш. $\Rightarrow \gamma_2$ нема уопштење $\xrightarrow{\gamma_2}$

$0 = 0$

$0 = 1$ ↯

$d = 0$

$a = 0$

$\Rightarrow \gamma_2$ одг. блоку 1×1

$\gamma_1 \rightsquigarrow \gamma_3 \rightsquigarrow \gamma_4$ $\quad \gamma_2$

γ_1: $(A-2E)\gamma_3 = \gamma_1$

$$\begin{matrix} 0 = 0 \\ 0 = 0 \\ d = 1 \\ a = 0 \end{matrix} \Rightarrow \gamma_3 = \begin{bmatrix} 0 \\ b \\ c \\ 1 \end{bmatrix} \xrightarrow{b=c=0} \gamma_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

γ_3: $(A-2E)\gamma_4 = \gamma_3$

$$\begin{matrix} 0 = 0 \\ 0 = 0 \\ d = 0 \\ a = 1 \end{matrix} \Rightarrow \gamma_4 = \begin{bmatrix} 1 \\ b \\ c \\ 0 \end{bmatrix} \xrightarrow{b=c=0} \gamma_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
>> A=[2 0 0 0; 0 2 0 0; 0 0 2 1; 1 0 0 2]

A =

     2     0     0     0
     0     2     0     0
     0     0     2     1
     1     0     0     2

>> [P D]=jordan(A)

P =

     0     0     1     0
     0     0     0     1
     1     0     0     0
     0     1     0     0


D =

     2     1     0     0
     0     2     1     0
     0     0     2     0
     0     0     0     2
```

$$J = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & \\ & \boxed{\;} \end{bmatrix} \rightsquigarrow P = \begin{bmatrix} \boxed{\gamma_1 \; \gamma_3 \; \gamma_4} & \boxed{\gamma_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D=\begin{bmatrix} \boxed{\begin{matrix}2&1&\\&2&1\\&&2\end{matrix}} & \\ & \boxed{2}\end{bmatrix} \rightsquigarrow P=\left[\boxed{\gamma_1\ \gamma_3\ \gamma_4}\ \boxed{\gamma_2}\right]=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$$

2	1	0	0
0	2	1	0
0	0	2	0
0	0	0	2

$e^{tD}=?\qquad D=\begin{bmatrix}B&\\&2\end{bmatrix}\Rightarrow e^{tD}=\begin{bmatrix}e^{tB}&\\&e^{2t}\end{bmatrix}$

$$e^{tB}=e^{t\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}+t\overbrace{\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}}^{=N}}\overset{(2)}{=}e^{t\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}}\cdot e^{tN}=e^{2t}\cdot E\cdot\left(E+\underline{tN}+\frac{t^2}{2}N^2\right)=e^{2t}\begin{bmatrix}1&t&\frac{t^2}{2}\\&1&t\\&&1\end{bmatrix}$$

$2E\cdot N=N\,2E$

$N^2=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$

$N^3=0$

$e^{tD}=e^{2t}\begin{bmatrix}1&t&\frac{t^2}{2}&\\&1&t&\\&&1&\\&&&1\end{bmatrix}$, ОР: $X(t)=P\cdot e^{tD}\cdot c,\ c\in\mathbb{R}^4$

② $X'=AX$

$A=\begin{bmatrix}2&1&1\\0&1&-1\\0&1&3\end{bmatrix}\qquad \lambda_1=\lambda_2=\lambda_3=2\Rightarrow k=3$

$(A-2E)\gamma=0\Rightarrow\gamma=a\cdot\underline{\begin{bmatrix}1\\0\\0\end{bmatrix}}+b\cdot\underline{\begin{bmatrix}0\\1\\-1\end{bmatrix}}\Rightarrow \dim\operatorname{Ker}(A-2E)=2\Rightarrow$ 2 блока $(m=2)$

$k=3, m=2$: једино $D=\begin{bmatrix}2&1&\\&2&\\&&2\end{bmatrix}$, $e^{tD}=\begin{bmatrix}e^{t\begin{bmatrix}2&1\\&2\end{bmatrix}}&\\&e^{2t}\end{bmatrix}=e^{2t}\begin{bmatrix}1&t&0\\0&1&0\\0&0&1\end{bmatrix}$.

$P=?\qquad \gamma_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \gamma_2=\begin{bmatrix}0\\1\\-1\end{bmatrix}$ ⤳ треба 1 уопштени ⤳ ~~да~~ ли је за γ_1 или γ_2?

$(A-2E)\gamma_3=\gamma_1$

$\begin{bmatrix}0&1&1\\0&-1&-1\\0&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$

$b+c=1$
$-b-c=0$
$b+c=0$
↯

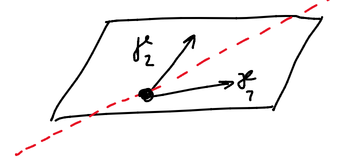
$(A-2E)\gamma_3=\gamma_2$

$b+c=0$
$-b-c=1$
$b+c=-1$
↯

γ_3 одговара овом потпростору $\dim=1$

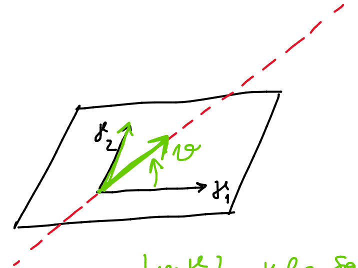
сопствени потпростор

сопствени потпростор
$= \mathcal{L}in\{\gamma_1, \gamma_2\}$



γ_3 одговара овом потпростору $\dim = 1$

имамо базу
од $\ker(A-2E)$:



$\{v, \gamma_2\}$ – нова база

$v = ?$, $v \in \mathcal{L}in\{\gamma_1, \gamma_2\} \Rightarrow v = \alpha \cdot \gamma_1 + \beta \cdot \gamma_2 \quad (\alpha, \beta = ?)$

$\rightarrow$ хоћемо да v има уопштени γ_3

$\exists \gamma_3$ тд: $(A-2E)\gamma_3 = v$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ -\beta \end{bmatrix}$$

$$\left.\begin{matrix} b + c = \alpha \\ -b - c = \beta \end{matrix}\right\} \Rightarrow \alpha = -\beta$$
$$b + c = -\beta$$

нпр. $\alpha = 1$, $\beta = -1$

$b + c = 1$, $b = 1$, $c = 0$, $a = 0$

$$\Rightarrow v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad и \quad \gamma_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

сопствени: v и γ_2

уопштени: γ_3

$$\Rightarrow P = \begin{bmatrix} v\downarrow & \gamma_3\downarrow & \gamma_2\downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

```
>> A=[2 1 1; 0 1 -1; 0 1 3]

A =

     2     1     1
     0     1    -1
     0     1     3

>> [P D]=jordan(A)

P =

     1     0     0
    -1     1    -1
     1     0     1


D =

     2     1     0
     0     2     0
     0     0     2
```

ОР: $X(t) = P \cdot e^{tD} \cdot c$, $c \in \mathbb{R}^4$

③ $X' = AX$

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 2$

$\lambda_{3/4} = 1 \pm i$

$$A=\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \lambda_{3/4}=1\pm i$$

$\underline{\lambda_1=\lambda_2=2 \ (k=2)}$

$$(A-2E)\gamma=0 \Rightarrow \gamma=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$\Rightarrow m=1 \Rightarrow \begin{bmatrix}2&1\\ &2\end{bmatrix}$$

$$(A-2E)\gamma_2=\gamma_1 \Rightarrow \gamma_2=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$\underline{\lambda_{3/4}=1\pm i}$

$$1+i \rightsquigarrow \begin{bmatrix}1&1\\-1&1\end{bmatrix}$$

$\alpha=1, \beta=1$

$$(A-(1+i)E)\gamma_3=0 \rightsquigarrow \gamma_3=\begin{bmatrix}1\\-2\\-1+i\\-2i\end{bmatrix}=\begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}+i\cdot\begin{bmatrix}0\\0\\1\\-2\end{bmatrix}$$

$$D=\begin{bmatrix}\boxed{\begin{matrix}2&1\\&2\end{matrix}} & \\ & \boxed{\begin{matrix}1&1\\-1&1\end{matrix}}\end{bmatrix} \rightsquigarrow e^{tD}=\begin{bmatrix}e^{t\begin{bmatrix}2&1\\0&2\end{bmatrix}} & \\ & e^{t\begin{bmatrix}1&1\\-1&1\end{bmatrix}}\end{bmatrix}=\begin{bmatrix}e^{2t}\cdot\begin{bmatrix}1&t\\0&1\end{bmatrix} & \\ & e^{t}\cdot R_t\end{bmatrix}=\begin{bmatrix}e^{2t} & t\cdot e^{2t} & 0 & 0\\ 0 & e^{2t} & 0 & 0\\ 0 & 0 & e^{t}\cos t & e^{t}\sin t\\ 0 & 0 & -e^{t}\sin t & e^{t}\cos t\end{bmatrix}$$

$$P=\begin{bmatrix}\gamma_1 & \gamma_2 & \operatorname{Re}\gamma_3 & \operatorname{Im}\gamma_3\end{bmatrix}=\begin{bmatrix}1&1&1&0\\0&0&-2&0\\0&1&-1&1\\0&0&0&-2\end{bmatrix}$$

$$\text{OP}: X(t)=P\cdot e^{tD}\cdot c,\ c\in\mathbb{R}^4$$

(4) $X'=AX$

$$A=\begin{bmatrix}-1&0&-1&0\\1&0&1&1\\2&0&1&0\\1&-1&0&0\end{bmatrix} \qquad \lambda_{1/2}=\lambda_{3/4}=\pm i$$

$$\lambda_1=i,\ k=2 \qquad 0+i\cdot 1 \rightsquigarrow \begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

$$D=\begin{bmatrix}\boxed{\begin{matrix}0&1\\-1&0\end{matrix}} & \boxed{\begin{matrix}0&0\\0&0\end{matrix}}\\ & \boxed{\begin{matrix}0&1\\-1&0\end{matrix}}\end{bmatrix} \swarrow m=2 \qquad \vee \qquad D=\boxed{\begin{bmatrix}0&1&1&0\\-1&0&0&1\\&&0&1\\&&-1&0\end{bmatrix}} \swarrow m=1$$

↳ 2 (комплексна) Жорд. бл. ↳ 1 (комплексни) Ж. бл.

$$(A-iE)\gamma_1=0$$

$$\therefore\ \gamma_1 = \begin{bmatrix} 0 \\ -i \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{} + i \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{} \quad \Rightarrow \quad \dim_{\mathbb{C}} Ker(A - iE) = \dim_{\mathbb{C}} Lin\{\gamma_1\} = 1 \Rightarrow m = 1$$

$$\Rightarrow 1 \text{ Ж. кл.}$$

2 реални век.

1 компл. век.

$$\Rightarrow D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}$$

ако би се добила 2 собс. век: γ_1, γ_2

$$m=2 \Rightarrow D = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} Re\,\gamma_1 & Im\,\gamma_1 & Re\,\gamma_2 & Im\,\gamma_2 \end{bmatrix}, \quad e^{tD} = \begin{bmatrix} R_t & \\ & R_t \end{bmatrix}$$

γ_2 - присъединен за γ_1: $(A - iE)\gamma_2 = \gamma_1$

$$\therefore\ \gamma_2 = \begin{bmatrix} 1 \\ 0 \\ -1-i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} Re\,\gamma_1 & Im\,\gamma_1 & Re\,\gamma_2 & Im\,\gamma_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$e^{tD} = ? \qquad D = \begin{bmatrix} R & E \\ O & R \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D = \underbrace{\begin{bmatrix} R & O \\ O & R \end{bmatrix}}_{J} + \underbrace{\begin{bmatrix} O & E \\ O & O \end{bmatrix}}_{N}$$

$$(J \not\sim E) \qquad J \cdot N = \begin{bmatrix} R & O \\ O & R \end{bmatrix} \begin{bmatrix} O & E \\ O & O \end{bmatrix} = \begin{bmatrix} O & R \\ O & O \end{bmatrix}$$

$$N \cdot J = \begin{bmatrix} O & E \\ O & O \end{bmatrix} \begin{bmatrix} R & O \\ O & R \end{bmatrix} = \begin{bmatrix} O & R \\ O & O \end{bmatrix}$$

$$\Rightarrow JN = NJ \Rightarrow e^{tD} \overset{(2)}{=} e^{tJ} \cdot e^{tN}$$

$$e^{tJ} = \begin{bmatrix} e^{tR} & \\ & e^{tR} \end{bmatrix} = \begin{bmatrix} R_t & \\ & R_t \end{bmatrix}$$

$$e^{tJ} = \begin{bmatrix} e^{tR} & \\ & e^{tR} \end{bmatrix} = \begin{bmatrix} R_t & \\ & R_t \end{bmatrix}$$

$$e^{tN} = \overset{4\times 4}{E} + t\cdot N = \begin{bmatrix} E & t\cdot E \\ 0 & E \end{bmatrix}$$

$$N^2 = \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{tD} = \begin{bmatrix} R_t & 0 \\ 0 & R_t \end{bmatrix} \cdot \begin{bmatrix} E & tE \\ 0 & E \end{bmatrix} = \begin{bmatrix} R_t & tR_t \\ 0 & R_t \end{bmatrix} =$$

$$= \begin{bmatrix} \cos t & \sin t & t\cos t & t\sin t \\ -\sin t & \cos t & -t\sin t & t\cos t \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{bmatrix}.$$

$$\text{ОР}: X(t) = P\cdot e^{tD}\cdot c, \quad c \in \mathbb{R}^4.$$

Враћа нам комплексну верзију Ж. форме (ми хоћемо реалну верзију)

```
>> A=[-1 0 -1 0; 1 0 1 1; 2 0 1 0; 1 -1 0 0]

A =

    -1     0    -1     0
     1     0     1     1
     2     0     1     0
     1    -1     0     0

>> [P D]=jordan(A)

P =

   0.0000 + 0.0000i   0.5000 - 0.5000i   0.0000 + 0.0000i   0.5000 + 0.5000i
   0.5000 + 0.5000i   0.0000 + 0.0000i   0.5000 - 0.5000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i
   0.5000 - 0.5000i   0.0000 + 0.0000i   0.5000 + 0.5000i   0.0000 + 0.0000i


D =

   0.0000 - 1.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i
```