(1) Нека је $A=[a_{ij}]_{i,j=1}^{n} \in M_n(\mathbb{R})$, $a_{ij} \geq 0, \forall i \neq j$. Нека је $B = e^A = [b_{ij}]_{i,j=1}^{n} \in M_n(\mathbb{R})$. Доказати $b_{ij} \geq 0, \forall i,j$.

$$A = \begin{bmatrix} ? & \geq 0 \\ \geq 0 & ? \end{bmatrix} \rightsquigarrow B = e^A = \begin{bmatrix} \geq 0 \end{bmatrix}$$

(*) ако $a_{ij} \geq 0\ \forall i,j \Rightarrow b_{ij} \geq 0\ \forall i,j$

пр. $$A = \begin{bmatrix} -3 & & \\ & 7 & \\ & & -11 \end{bmatrix} = \underbrace{\begin{bmatrix} -11 & & \\ & -11 & \\ & & -11 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 8 & & \\ & 18 & \\ & & 0 \end{bmatrix}}_{Y} \rightsquigarrow e^A = e^X \cdot e^Y$$

$$A = X + Y$$

$$X = -\underbrace{\max_{1 \leq i \leq n} |a_{ii}|}_{M} \cdot E = -ME$$

(или: $X = \min(\{a_{ii} \mid a_{ii} < 0, 1 \leq i \leq n\} \cup \{0\})$).

$$Y = A - X = A + ME = \begin{cases} \overset{\geq 0}{a_{ij}}, & i \neq j \\ \overset{\geq 0}{a_{ii} + M} \end{cases} = [\geq 0]$$

$XY = YX$? $\qquad -ME \cdot Y = -MY = Y(-M) = Y(-ME) = YX$

(2)
$$\Rightarrow e^A = e^{X+Y} = e^X \cdot e^Y = e^{-ME} \cdot e^Y = e^{-M} \cdot E \cdot e^Y = \underbrace{e^{-M}}_{\geq 0} \cdot e^Y \quad = [\geq 0]$$

сви ел. ≥ 0

Решавање ЛСДЈКК свођењем на Жорданову форму

$X' = AX$, $A \in M_n(\mathbb{R})$

ОР: $X(t) = e^{tA} \cdot c$, $c \in \mathbb{R}^n$

$A = P \cdot D \cdot P^{-1}$

↳ у Жордановој нормалној форми $(A \sim D)$

P - матрица преласка

$e^{tA} = e^{tPDP^{-1}} \overset{(7)}{=} P e^{tD} P^{-1}$

↳ знамо да израчунамо

$$e^{tA} = e^{tPDP^{-1}} \overset{(7)}{=} P\,\underbrace{e^{tD}}_{\text{знамо да израчунамо}}P^{-1}$$

(2) $x_1' = x_1 - x_2 + x_3$

$x_2' = x_1 + x_2 - x_3$

$x_3' = 2x_1 - x_2$

а) Наћи ОР.

б) Наћи ПР $X(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

а) $X' = AX$, $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

$$\det(A - \lambda E) = 0$$

$$0 = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda)^2 + \underline{2-1} - 2(1-\lambda) - \underline{\lambda} - (1-\lambda) = (1-\lambda)(-\lambda+\lambda^2+1-2-1) = (1-\lambda)(\lambda^2-\lambda-2) = (1-\lambda)(\lambda-2)(\lambda+1)$$

$\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 1$ → реалне и различите: D-дијагонална
P- 3 сопс. век

$$D = \begin{bmatrix} -1 & & \\ & 2 & \\ & & 1 \end{bmatrix} \Rightarrow e^{tD} = \begin{bmatrix} e^{-t} & & \\ & e^{2t} & \\ & & e^{t} \end{bmatrix}$$

```
>> A=[1 -1 1; 1 1 -1; 2 -1 0]

A =

     1    -1     1
     1     1    -1
     2    -1     0

>> eig(A)

ans =

   -1.0000
    1.0000
    2.0000
```

$\lambda_1 = -1$: $(A - \lambda_1 E)\gamma_1 = 0$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2a - b + c = 0$

$a + 2b - c = 0$

$2a - b + c = 0$

$\vdots$

$b = -3a$

$c = -5a$

$$\begin{bmatrix} a \\ -3a \\ -5a \end{bmatrix} = a \cdot \underbrace{\begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}}_{\gamma_1}, \quad a = 1$$

$\lambda_2 = 2$: $\gamma_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\qquad$ $\lambda_3 = 1$: $\gamma_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

```
P =
        γ1        γ3        γ2
    0.1690   -0.5774    0.7071
   -0.5071   -0.5774    0.0000
   -0.8452   -0.5774    0.7071

D =

   -1.0000         0         0
         0    1.0000         0
         0         0    2.0000
```

$$P = \begin{bmatrix} \gamma_1\downarrow & \gamma_2\downarrow & \gamma_3\downarrow \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 1 \\ -5 & 1 & 1 \end{bmatrix}$$
$$\begin{array}{ccc} \gamma_1 & \gamma_2 & \gamma_3 \end{array}$$

$P^{-1} = ?$

$$P^{-1} = \frac{1}{\det P}\cdot \operatorname{Adj} P = \frac{1}{-6}\cdot \begin{bmatrix} +\begin{vmatrix}0&1\\1&1\end{vmatrix} & -\begin{vmatrix}-3&1\\-5&1\end{vmatrix} & +\begin{vmatrix}-3&0\\-5&1\end{vmatrix} \\ -\begin{vmatrix}1&1\\1&1\end{vmatrix} & +\begin{vmatrix}1&1\\-5&1\end{vmatrix} & -\begin{vmatrix}1&1\\-5&1\end{vmatrix} \\ +\begin{vmatrix}1&1\\0&1\end{vmatrix} & -\begin{vmatrix}1&1\\-3&1\end{vmatrix} & +\begin{vmatrix}1&1\\-3&0\end{vmatrix} \end{bmatrix}^T = \frac{1}{-6}\begin{bmatrix} -1 & -2 & -3 \\ 0 & 6 & -6 \\ 1 & -4 & 3 \end{bmatrix}^T$$

$$= \frac{1}{6}\begin{bmatrix} 1 & 0 & -1 \\ 2 & -6 & 4 \\ 3 & 6 & -3 \end{bmatrix}$$

```
>> P = [1 1 1; -3 0 1; -5 1 1]

P =

     1     1     1
    -3     0     1
    -5     1     1

>> inv(P)

ans =

    0.1667   -0.0000   -0.1667
    0.3333   -1.0000    0.6667
    0.5000    1.0000   -0.5000

>> det(P)

ans =

    -6
```

OP: $x(t) = e^{tA}\cdot c = P\cdot e^{tD}\cdot \underbrace{P^{-1}\cdot c}_{c_1} = P\cdot e^{tD}\cdot c_1$

$c, c_1 \in \mathbb{R}^3$

```
>> A=[1 -1 1; 1 1 -1; 2 -1 0]

A =

     1    -1     1
     1     1    -1
     2    -1     0

>> syms t
>> expm(A*t)

ans =

[     exp(-t)/6 + exp(2*t)/3 + exp(t)/2, exp(t) - exp(2*t),     (2*exp(2*t))/3 - exp(-t)/6 - exp(t)/2]
[                  exp(t)/2 - exp(-t)/2,            exp(t),                    exp(-t)/2 - exp(t)/2]
[ exp(2*t)/3 - (5*exp(-t))/6 + exp(t)/2, exp(t) - exp(2*t), (5*exp(-t))/6 + (2*exp(2*t))/3 - exp(t)/2]
```

b) $x(t) = Pe^{tD}\cdot c_1$

$x(0) = P\cdot c_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow c_1 = P^{-1}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = \dots$

$x(t) = Pe^{tD}P^{-1}c$

$x(0) = PP^{-1}\cdot c = c = \begin{bmatrix}1\\1\end{bmatrix} \Rightarrow x(t) = e^{tA}\cdot x(0)$

③ $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

$\det(A-\lambda E) = 0$

$\lambda_1 = -3$

$\lambda_{2/3} = 2 \pm i$

```
>> A=[-3 0 0; 0 3 -2; 0 1 1]

A =

    -3     0     0
     0     3    -2
     0     1     1

>> eig(A)

ans =

   2.0000 + 1.0000i
   2.0000 - 1.0000i
  -3.0000 + 0.0000i
```

$\lambda_1 = -3$ $\quad\lambda_{2/3} = 2 \pm i$

$$D = \begin{bmatrix} \boxed{-3} & & \\ & \boxed{\begin{matrix} 2 & 1 \\ -1 & 2 \end{matrix}}_{2\times 2} \end{bmatrix}$$

$\alpha + i\beta \leftrightarrow \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

$\lambda_1 = -3 \rightsquigarrow (A-\lambda_1 E)v_1 = 0 \rightsquigarrow v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$\underline{\lambda_2 = 2+i} \rightsquigarrow (A-\lambda_2 E)v_2 = 0$

$$\begin{bmatrix} -3-(2+i) & 0 & 0 \\ 0 & 3-(2+i) & -2 \\ 0 & 1 & 1-(2+i) \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 0 & 3-(2+i) & -2 \\ 0 & 1 & 1-(2+i) \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \vec{0}$$

$a, b, c \in \mathbb{C}$

$$\left.\begin{aligned} (-5-i)a &= 0 \\ (1-i)b - 2c &= 0 \\ b + (-1-i)c &= 0 \end{aligned}\right\} \longrightarrow / \cdot \frac{(1+i)}{2} \Rightarrow b-(1+i)c=0$$

$a = 0$

$b = (1+i)c$

тј. $c=1$: $\gamma_2 = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{Re\gamma_2} + i \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{Im\gamma_2}$

$$P = \begin{bmatrix} \gamma_1 \downarrow & Re\gamma_2 \downarrow & Im\gamma_2 \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

```
>> [P D]=eig(A)

P =
          γ2                     γ3                    γ1
   0.0000 + 0.0000i     0.0000 + 0.0000i     1.0000 + 0.0000i
   0.8165 + 0.0000i     0.8165 + 0.0000i     0.0000 + 0.0000i
   0.4082 - 0.4082i     0.4082 + 0.4082i     0.0000 + 0.0000i

D =
          λ2
   2.0000 + 1.0000i     0.0000 + 0.0000i     0.0000 + 0.0000i
   0.0000 + 0.0000i     2.0000 - 1.0000i λ3  0.0000 + 0.0000i λ1
   0.0000 + 0.0000i     0.0000 + 0.0000i    -3.0000 + 0.0000i
```

(γ_3 не треба тражити: $\lambda_3 = \overline{\lambda_2} \Rightarrow \gamma_3 = \overline{\gamma_2}$)

$e^{tD} = ?$

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \Rightarrow B^k = \begin{bmatrix} B_1^k & \\ & B_2^k \end{bmatrix} \Rightarrow e^{tB} = \begin{bmatrix} e^{tB_1} & \\ & e^{tB_2} \end{bmatrix}$$

↳ блоковска матрица

$$D = \begin{bmatrix} \boxed{-3}_{1\times1} & \\ & \boxed{\begin{matrix} 2 & 1 \\ -1 & 2 \end{matrix}}_{2\times2} \end{bmatrix} \rightsquigarrow e^{tD} = \begin{bmatrix} \boxed{e^{t[-3]}}_{1\times1} & \\ & \boxed{e^{t\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}}_{2\times2} \end{bmatrix} = \begin{bmatrix} e^{-3t} & & \\ & e^{2t}\cos t & e^{2t}\sin t \\ & -e^{2t}\sin t & e^{2t}\cos t \end{bmatrix}$$

Подсетник: $e^{t\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}} = e^{t\alpha} \cdot R_{t\beta}$

ОР: $x(t) = P e^{tD} \cdot c$, $c \in \mathbb{R}^3$

Жорданова нормална форма: $\begin{bmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & \\ & & \ddots & \\ & & & \boxed{B_k} \end{bmatrix}$

$B_i = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$ $\vee$ $B_i = \begin{bmatrix} R & E & & \\ & R & E & \\ & & \ddots & E \\ & & & R \end{bmatrix}$

када имамо поновљену $\lambda \in \mathbb{R}$

када имамо поновљену $\alpha + i\beta \in \mathbb{C}$

$R = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

поновљену $\lambda \in \mathbb{K}$

$$R=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix},\quad E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

нпр. $n=4$

$\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda$

$A = P\cdot D\cdot P^{-1}$

D може бити:

$$\begin{bmatrix}\boxed{\lambda} & & & \\ & \boxed{\lambda} & & \\ & & \boxed{\lambda} & \\ & & & \boxed{\lambda}\end{bmatrix},\ \begin{bmatrix}\boxed{\lambda} & & & \\ & \lambda & 1 & \\ & & \lambda & 1\\ & & & \lambda\end{bmatrix},\ \begin{bmatrix}\lambda & 1 & & \\ & \lambda & & \\ & & \lambda & 1\\ & & & \lambda\end{bmatrix},\ \begin{bmatrix}\lambda & 1 & & \\ & \lambda & & \\ & & \boxed{\lambda} & \\ & & & \boxed{\lambda}\end{bmatrix},\ \begin{bmatrix}\lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1\\ & & & \lambda\end{bmatrix}$$

4) $A=\begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 1 & 0\\ 0 & 0 & 2 & 0\\ 1 & 0 & 0 & 2\end{bmatrix}$

$\lambda_1=\lambda_2=\lambda_3=\lambda_4=2$

$k=4$ – алгебарска вишеструкост

$(A-2E)\gamma=0$

$$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\vec{0} \qquad \begin{matrix}0=0\\ c=0\\ 0=0\\ a=0\end{matrix} \quad\Rightarrow\quad \begin{bmatrix}0\\ b\\ 0\\ d\end{bmatrix}=b\cdot\begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix}+d\cdot\begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}$$

$\dim(\mathrm{Ker}(A-2E))=2 \Rightarrow m=2$ – геометријска вишеструкост

→ имамо 2 Жорданова блока

$$\begin{bmatrix}2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2\end{bmatrix} \quad \vee \quad \begin{bmatrix}2 & 1 & & \\ & 2 & & \\ & & 2 & 1\\ & & & 2\end{bmatrix}$$

минимални полином? $\mu(\lambda)$

$\varphi(\lambda)=(\lambda-2)^4=\det(A-\lambda E)$

$\mu \mid \varphi \Rightarrow \mu(\lambda)=(\lambda-2)^{\ell},\ \ell\in\{1,2,3,4\}$

$\mu(A)=0$

$$(A-2E)^1=\begin{bmatrix} & & & \\ & & 1 & \\ & & & \\ 1 & & & \end{bmatrix}\neq 0$$

$$(A-2E)^2=\begin{bmatrix} & & & \\ & & 1 & \\ & & & \\ 1 & & & \end{bmatrix}\begin{bmatrix} & & & \\ & & 1 & \\ & & & \\ 1 & & & \end{bmatrix}=O \Rightarrow \mu(\lambda)=(\lambda-2)^2 \Rightarrow \deg\mu=2$$

$\Downarrow$

2 је ред највећег Жордановог блока

$$\Rightarrow D=\begin{bmatrix}2 & 1 & & \\ & 2 & & \\ & & 2 & 1\\ & & & 2\end{bmatrix}$$

γ_1 ? γ_3 ?

генералисани сопс. век. (утопљени)

$\gamma_1 \longrightarrow \gamma_2$

$\gamma_3 \longrightarrow \gamma_4$

$$\begin{bmatrix}0\\ b\\ 0\\ d\end{bmatrix}=b\cdot\underbrace{\begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix}}_{\gamma_1}+d\cdot\underbrace{\begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}}_{\gamma_3}$$

$\gamma_1 \rightsquigarrow \gamma_2$: $(A-2E)\gamma_2 = \gamma_1$

$$\begin{aligned} 0 &= 0 \\ c &= 1 \\ 0 &= 0 \\ a &= 0 \end{aligned} \qquad \gamma_2 = \begin{bmatrix} 0 \\ b \\ 1 \\ d \end{bmatrix} \xrightarrow{b=d=0} \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

```
>> A=[2 0 0 0; 0 2 1 0; 0 0 2 0; 1 0 0 2]

A =

     2     0     0     0
     0     2     1     0
     0     0     2     0
     1     0     0     2

>> [P D]=eig(A)

P =

         0    0.0000         0         0
         0         0    1.0000   -1.0000
         0         0         0    0.0000
    1.0000   -1.0000         0         0


D =

     2     0     0     0
     0     2     0     0
     0     0     2     0
     0     0     0     2
```

$\gamma_3 \rightsquigarrow \gamma_4$: $(A-2E)\gamma_4 = \gamma_3 \Rightarrow \gamma_4 = \begin{bmatrix} 1 \\ b \\ 0 \\ d \end{bmatrix} \xrightarrow{b=d=0} \gamma_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (\det P \neq 0) \qquad (\rightsquigarrow \ \gamma_1\gamma_2\gamma_3\gamma_4 \ \rightsquigarrow \ \gamma_3\gamma_4\gamma_1\gamma_2)$$

$$e^{tD} = ? \qquad e^{tD} = \begin{bmatrix} e^{t\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}} & \\ & e^{t\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}} \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & t \\ & & & 1 \end{bmatrix}.$$

```
>> jordan(A)

ans =

     2     1     0     0
     0     2     0     0
     0     0     2     1
     0     0     0     2

>> [P D]=jordan(A)

P =

     0     1     0     0
     0     0     1     0
     0     0     0     1
     1     0     0     0


D =

     2     1     0     0
     0     2     0     0
     0     0     2     1
     0     0     0     2
```

$$e^{t\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}} = e^{t\begin{bmatrix} 2 & \\ & 2 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} \overset{(2)}{=} e^{t\begin{bmatrix} 2 & \\ & 2 \end{bmatrix}} \cdot e^{t\overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{\text{"}N}} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \cdot \big(\underbrace{E}_{k=0} + \underbrace{tN}_{k=1}\big) = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$2E \cdot N = N \cdot 2E$

$$N^2 = \begin{bmatrix} & 1 \\ & \end{bmatrix} \begin{bmatrix} & 1 \\ & \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$N^k = 0, \ k \geq 2$

OŘ: $X(t) = P \cdot e^{tD} \cdot c, \ c \in \mathbb{R}^4$