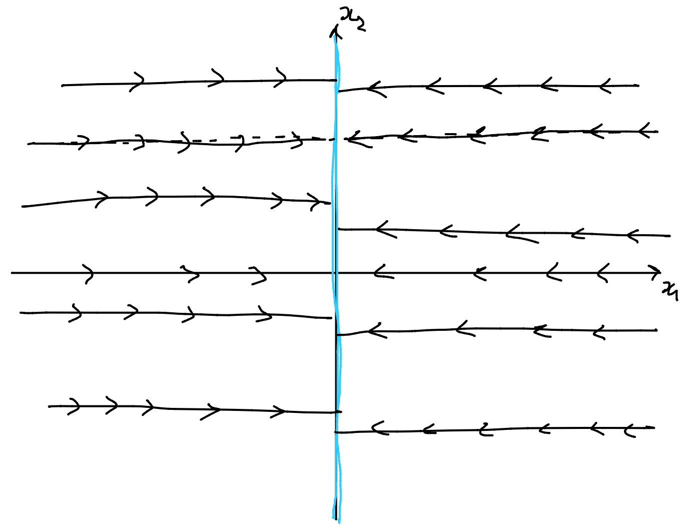
4. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

(1) $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$; (2) $A = \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}$; (3) $A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$.

(4.2) $X^* = ?$

$$\left.\begin{aligned} -5x_1 &= 0 \\ 0 &= 0 \end{aligned}\right\} x_1 = 0 \quad , \quad X^* \in \{(0, \lambda) \mid \lambda \in \mathbb{R}\}$$

↳ неизоловани еквилибријуми




$x_1' = -5x_1$
$x_2' = 0$

$x_2 = c_2$
$x_1 = c_1 e^{-5t}$ , $c_1, c_2 \in \mathbb{R}$

$c_1 > 0$: $\quad c_1 e^{-5t} > 0$

$t \to -\infty \quad t = 0 \quad t \to \infty$

$+\infty \longrightarrow c_1 \longrightarrow 0$

$c_1 < 0$: $\quad c_1 e^{-5t} < 0$

$t \to -\infty \quad t = 0 \quad t \to +\infty$

$-\infty \longrightarrow c_1 \longrightarrow 0$

## Експонент матрице

$A \in M_n(\mathbb{R})$ ↙ $\mathbb{C}$

$\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$X' = AX$ , $A \in M_n(\mathbb{R})$
$X(t), X'(t) \in \mathbb{R}^n$

$$\begin{aligned} x_1' &= a_{11}x_1 + \dots + a_{1n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \dots + a_{nn}x_n \end{aligned}$$

(ИСДЈКК)

ОР: $X(t) = \underbrace{e^{tA}} \cdot c$ , $c \in \mathbb{R}^n$ , $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\cancel{c \cdot e^{tA}}$ $(n\times 1)\times(n\times n)$

**$\boxed{\text{Тврђење 52.}}$ (Својства експонента.)**

(1) $e^{\mathbf{0}} = \mathrm{Id}$; O – нула матрица $\quad \mathrm{Id} = id = I = E \in M_n(\mathbb{R})$
(2) $AB = BA \Rightarrow e^{A+B} = e^A e^B$;
(3) $AB = BA \Rightarrow Be^A = e^A B$; → стеју: $Ae^A = e^A A$
(4) $e^A = \lim_{n\to\infty} (\mathrm{Id} + \frac{A}{n})^n$;

**Тврђење 52.** (Својства експонента.)

(1) $e^0 = \mathrm{Id}$; 0 – нула матрица $\quad \mathrm{Id} = id = I = E \in M_n(\mathbb{R})$

(2) $AB = BA \Rightarrow e^{A+B} = e^A e^B$;

(3) $AB = BA \Rightarrow Be^A = e^A B$; $\longrightarrow$ специ. $Ae^A = e^A A$

(4) $e^A = \lim\limits_{n\to\infty} (\mathrm{Id} + \frac{A}{n})^n$;

(5) за $U = \mathbb{R}^n$, тј. $A \in M_n(\mathbb{R})$ важи $\frac{d}{dt} e^{tA} = e^{tA} A = A e^{tA}$;

(6) за $U = \mathbb{R}^n$ важи $\det e^A = e^{\operatorname{tr} A}$;

(7) за $U = \mathbb{R}^n$ важи $e^{P^{-1}AP} = P^{-1} e^A P$.

① Решити систем ДЈ $X' = AX$, одређивањем $e^{tA}$ у облику степеног реда, ако је:

а) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ б) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ в) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ г) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in \mathbb{R}$

домаћи (исто као а)

$X(t) = e^{tA} \cdot c, \ c \in \mathbb{R}^2$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \qquad , A^k = ?$$

а) $A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = A^2 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \rightsquigarrow A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \leftarrow$ индукцијом

Б: $k=1$: $A^1 = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ✓

Х: $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

К: $A^{k+1} = ?$

$$A^{k+1} = A^k \cdot A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{t^k}{k!} & \sum\limits_{k=0}^{\infty} \frac{t^k}{k!} \cdot k \\ 0 & \sum\limits_{k=0}^{\infty} \frac{t^k}{k!} \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \qquad , \sum_{k=0}^{\infty} \frac{t^k}{k!} k = \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!} = t \cdot \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} = t \cdot \underbrace{\sum_{k=0}^{\infty} \frac{t^k}{k!}} = t \cdot e^t$$

ОР: $X(t) = e^{tA}\cdot C = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1e^t + c_2te^t \\ c_2e^t \end{bmatrix}$, $c_1, c_2 \in \mathbb{R}$

в) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -E$$

$$A^3 = A^2A = -E\cdot A = -A$$

$$A^4 = A^3A = -A\cdot A = -A^2 = -(-E) = E$$

$$A^k = A^{4t+r} = \begin{cases} A, & r=1 \\ -E, & r=2 \\ -A, & r=3 \\ E, & r=0 \end{cases} = \begin{cases} (-1)^{\frac{r-1}{2}}\cdot A, & 2\nmid r \\ (-1)^{\frac{r}{2}}\cdot E, & 2\mid r \end{cases} = \begin{cases} (-1)^{\frac{k-1}{2}} A, & 2\nmid k \\ (-1)^{\frac{k}{2}}\cdot E, & 2\mid k \end{cases} = \begin{cases} (-1)^{l}\cdot A, & k=2l+1 \\ (-1)^{l}\cdot E, & k=2l \end{cases}$$

$k = 4t+r$

$$e^{tA} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\cdot A^k = \underbrace{\overbrace{\sum_{l=0}^{\infty}\frac{t^{2l+1}}{(2l+1)!}\cdot(-1)^l}^{\sin t}\cdot A}_{k=2l+1} + \underbrace{\overbrace{\sum_{l=0}^{\infty}\frac{t^{2l}}{(2l)!}\cdot(-1)^l}^{\cos t}\cdot E}_{k=2l} = \sin t\cdot A + \cos t\cdot E = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

ОР: $X(t) = e^{tA}\cdot C$, $C \in \mathbb{R}^2$

г) другачије од предавања: (завршите сами)

из дела в)

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\cdot\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = aE + bB$$

$(\lambda_{1/2} = a \pm ib)$

$$e^{tA} = e^{t(aE+bB)} = e^{taE + tbB} \overset{(2)}{=} e^{taE}\cdot e^{tbB}$$

да ли $taE$ и $tbB$ комутирају? ✓

из в)

$$e^{tbB} = \begin{bmatrix} \cos(tb) & \sin(tb) \\ -\sin(tb) & \cos(tb) \end{bmatrix} = R_{tb}$$

матрица ротације

додатни: пример да (2) не важи увек, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$.

② Испитати да ли постоје $A\in M_2(\mathbb{R})$ тд:

а) $e^A=\begin{bmatrix}1&0\\0&-4\end{bmatrix}$ б) $e^A=\begin{bmatrix}-1&0\\0&-4\end{bmatrix}$

у $\mathbb{R}$: $e^a=1$, $e^a=-1$

а) $e^A=\begin{bmatrix}1&0\\0&-4\end{bmatrix}$ / det

$$\det(e^A)=-4$$

$$\overset{(6)}{\Rightarrow} e^{\operatorname{tr}A}=-4<0 \ \lightning \ \Rightarrow \nexists A$$

б) $e^A=\begin{bmatrix}-1&0\\0&-4\end{bmatrix}$

$$\det\left(\begin{bmatrix}-1&0\\0&-4\end{bmatrix}\right)=4>0 \Rightarrow \operatorname{tr}A=\ln 4 \dots$$

$$\overset{(3)}{\underset{A\subset B}{\Rightarrow}} Ae^A=e^AA \qquad , A=\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}, \alpha+\delta=\ln 4$$

$$\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}\begin{bmatrix}-1&0\\0&-4\end{bmatrix}=\begin{bmatrix}-1&0\\0&-4\end{bmatrix}\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}$$

$$\begin{bmatrix}-\alpha&-4\beta\\-\gamma&-4\delta\end{bmatrix}=\begin{bmatrix}-\alpha&-\beta\\-4\gamma&-4\delta\end{bmatrix}$$

$$\left.\begin{matrix}\cancel{-\alpha=-\alpha}\\-4\beta=-\beta\\-\gamma=-4\gamma\\\cancel{-4\delta=-4\delta}\end{matrix}\right\} \beta=\gamma=0 \Rightarrow A=\begin{bmatrix}\alpha&0\\0&\delta\end{bmatrix} \rightsquigarrow A^k=\begin{bmatrix}\alpha^k&0\\0&\delta^k\end{bmatrix}$$

$$e^A=\sum_{k=0}^{\infty}\frac{A^k}{k!}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{\alpha^k}{k!}&0\\0&\frac{\delta^k}{k!}\end{bmatrix}=\begin{bmatrix}e^\alpha&0\\0&e^\delta\end{bmatrix}=\begin{bmatrix}-1&0\\0&-4\end{bmatrix} \ \lightning$$

$$A=\operatorname{diag}\{\lambda_1,\dots,\lambda_n\} \Rightarrow e^A=\operatorname{diag}\{e^{\lambda_1},\dots,e^{\lambda_n}\}$$

③ $\lambda\in\mathbb{C}$ соп. вр. од $A \Rightarrow e^\lambda$ соп. вр. од $e^A$

I) дато: $Av=\lambda v$, $v\neq 0$

доказати: $e^Av=e^\lambda v$

$$(\lambda E)^k=\lambda^kE^k=\lambda^kE$$

II) $\det(A-\lambda E)=0$

II) $\det(A-\lambda E)=0$

хоћемо да је 0

$$\det(e^A - e^\lambda E) = \det\left(\sum_{k=0}^{\infty}\frac{A^k}{k!} - \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\cdot E\right) = \det\left(\sum_{k=0}^{\infty}\frac{A^k-(\lambda E)^k}{k!}\right) =$$

$$= \det\left(\lim_{N\to\infty}\sum_{k=0}^{N}\frac{A^k-(\lambda E)^k}{k!}\right) =$$

$$\overset{(*)}{=} \lim_{N\to\infty}\det\left(\sum_{k=0}^{N}\frac{A^k-(\lambda E)^k}{k!}\right) =$$

$$\overset{(*)}{=} \lim_{N\to\infty}\det\Big(\underbrace{\sum_{k=0}^{N}\frac{(A-\lambda E)(A^{k-1}+\lambda A^{k-2}+\dots+\lambda^{k-1}E)}{k!}}_{=B_N}\Big) =$$

$$= \lim_{N\to\infty}\det\big((A-\lambda E)\cdot B_N\big) =$$

$$\overset{(*)}{=} \lim_{N\to\infty}\big(\underbrace{\det(A-\lambda E)}_{=0}\cdot\det B_N\big) =$$

$$= \lim_{N\to\infty} 0 = 0.$$

(*) $\det : M_n(\mathbb{R}) \to \mathbb{R}$ непр.

$A_N := \sum_{k=0}^{N}\frac{A^k-(\lambda E)^k}{k!} \longrightarrow A_\infty = e^A - e^\lambda E$

$\lim_{N\to\infty}\det A_N = \det\left(\lim_{N\to\infty} A_N\right) = \det(A_\infty)$

(*) $A^k - B^k = (A-B)(A^{k-1}+A^{k-2}B+\dots+B^{k-1})$

$A, B$ комутирају

$(A\cdot\lambda E = \lambda AE = \lambda A = \lambda E\cdot A)$

(*) $\det(AB) = \det A\cdot\det B$

4) $A = \begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$. Наћи $\det\left(e^{e^A}\right)$.

$A, e^A, e^{e^A} \in M_3(\mathbb{R})$

$$d = \det\left(e^{e^A}\right) \overset{(6)}{=} e^{\operatorname{tr} e^A}.$$

$e^A = ?$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{E} + \underbrace{\begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}}_{B}$$

$EB = B = BE$

$\Downarrow$ (2)

$e^A = e^{E+B} = e^E\cdot e^B$

$$e^E = e^{\operatorname{diag}\{1,1,1\}} = \operatorname{diag}\{e,e,e\} = eE$$

$e^B = ?$

$$B^2 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$B^3 = \begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix} = B$$

$$B^k = \begin{cases} B^2, & k=2l,\ l\geq 1\\ B, & k=2l+1,\ l\geq 0\end{cases}$$

$$e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!} = \underset{\substack{\uparrow \\ k=0}}{E} + \sum_{\ell=0}^{\infty} \frac{B}{(2\ell+1)!} + \sum_{\ell=1}^{\infty} \frac{B^2}{(2\ell)!} = E + B(\operatorname{sh}1) + B^2(\operatorname{ch}1 - 1) = \begin{bmatrix} \operatorname{ch}1 & 0 & \operatorname{sh}1 \\ 0 & 1 & 0 \\ \operatorname{sh}1 & 0 & \operatorname{ch}1 \end{bmatrix}$$

$$\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)!} = \operatorname{sh}1$$

$$\sum_{\ell=1}^{\infty} \frac{1}{(2\ell)!} = \operatorname{ch}1 - 1$$

$$\frac{e+e^{-1}}{2} = \text{парни} = \operatorname{ch}1$$

$$\frac{e-e^{-1}}{2} = \text{непарни} = \operatorname{sh}1$$

$$e^A = e^B \cdot e^E = eE \cdot e^B = e \cdot e^B$$

$$\operatorname{tr}(e^A) = e \cdot \operatorname{tr}(e^B) = e(2\operatorname{ch}1 + 1) = e\left(e + e^{-1} + 1\right) = e^2 + e + 1 \Rightarrow d = e^{e^2+e+1}.$$