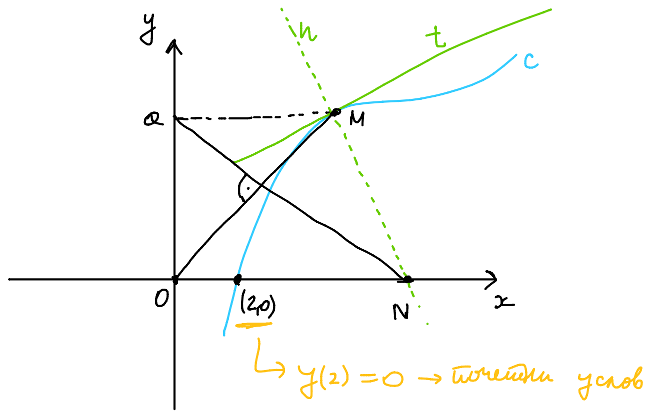
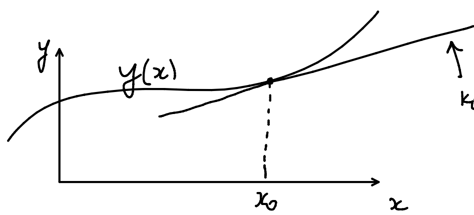
(1) Тачка M криве c се пројектује на y-осу у Q, а нормала на c у M сече x-осу у N. Наћи c ако важи $QN \perp MO$ $(\forall M \in c)$ и c пролази кроз (2,0). (O-коорд. почетак)

$c: y(x)$



↳ $y(2)=0$ → почетни услов

Где је y'?



коеф. правца: $\boxed{y'(x_0)}$

$M(x_0, y(x_0))$ $(M \in c)$

$\left.\begin{array}{l} x_Q = 0 \\ y_Q = y_M \end{array}\right\}$ $Q(0, y(x_0))$

$O(0,0)$

$N(x_N, 0)$

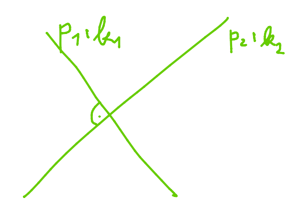
$t: \ k_t = y'(x_0)$

$n: \ n \perp t$

$k_n = -\dfrac{1}{y'(x_0)}$

$M, N \in n$

$$k_n = \frac{y_M - y_N}{x_M - x_N} = \frac{y(x_0) - 0}{x_0 - x_N}$$



$p_1 \perp p_2 \iff k_1 k_2 = -1 \iff k_2 = -\dfrac{1}{k_1}$

$$-y'(x_0)\, y(x_0) = x_0 - x_N \Rightarrow x_N = x_0 + y(x_0) \cdot y'(x_0)$$

$MO \perp QN$:

$$p_1 = MO: \ k_1 = \frac{y_M - y_O}{x_M - x_O} = \frac{y(x_0) - 0}{x_0 - 0} = \frac{y(x_0)}{x_0}$$

$$p_2 = QN: \ k_2 = \frac{y_N - y_Q}{x_N - x_Q} = \frac{0 - y(x_0)}{x_0 + y(x_0)y'(x_0) - 0} = -\frac{y(x_0)}{x_0 + y(x_0)y'(x_0)}$$

$$p_2 = QN: \quad k_2 = \frac{y_N - y_Q}{x_N - x_Q} = \frac{0 - y(x_0)}{x_0 + y(x_0)y'(x_0) - 0} = -\frac{y(x_0)}{x_0 + y(x_0)y'(x_0)}$$

$$p_1 \perp p_2 \Rightarrow k_1 k_2 = -1 \Rightarrow \frac{y(x_0)^2}{x_0^2 + x_0 y(x_0) y'(x_0)} = 1$$

$$x_0 y(x_0) y'(x_0) + x_0^2 = y(x_0)^2$$

$\forall x_0$ (локално)

правимо ДЈ:

$x_0 \rightsquigarrow x$

$y(x_0) \rightsquigarrow y(x) = y$

$y'(x_0) \rightsquigarrow y'(x) = y'$

$$\Rightarrow x\boxed{yy'} + x^2 = \boxed{y^2}, \quad y(2) = 0$$

1) $/ : y^2$

$$\frac{x}{y} \cdot y' + \left(\frac{x}{y}\right)^2 = 1 \quad (\text{ХОМ})$$

$$z = \frac{y}{x}, \quad z(2) = 0 \ \ldots$$

2) $z = y^2$

$z' = 2yy'$

$$x\frac{z'}{2} + x^2 = z, \quad z(2) = 0 \ \ldots \quad (\text{ЛИН})$$

7 ДЈ са тоталним диференцијалом

$$M(t,x)\,dt + N(t,x)\,dx = 0$$

$\exists F$, $dF(t,x) = M(t,x)\,dt + N(t,x)\,dx \Rightarrow$ ОР: $F = \text{const}$

$$\left[x' = \frac{f}{g} \Leftrightarrow x'g = f \Leftrightarrow \frac{dx}{dt} g - f = 0 \Leftrightarrow \underline{g\,dx - f\,dt = 0} \right]$$

дј у диференцијалном облику

потребан услов: $M'_x = N'_t$

довољан: ако радимо на просто-повезаној (ПП) области

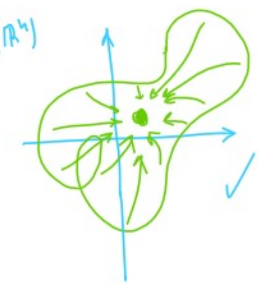
где су M,N деф, глатке или где посматрамо дј...

Просто-повезане (ПП) области

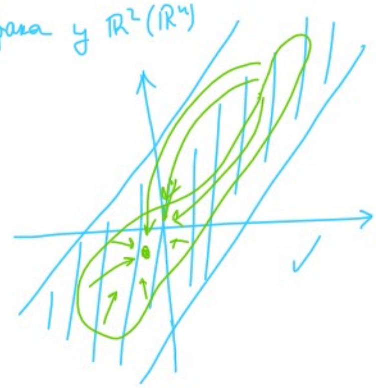
$(\pi_1(D) \cong 1)$

$D \subseteq \mathbb{R}^k$ је ПП $\Leftrightarrow$ сваку непрекидну затворену криву у D можемо скупити (непрекидно) у тачку кроз D и D је повезана $(\pi_0(D) \cong \mathbb{Z})$

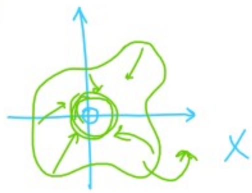
Пр. 1) $\mathbb{R}^2$ $(\mathbb{R}^n)$



2) трака у $\mathbb{R}^2$ $(\mathbb{R}^n)$



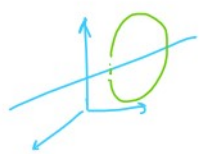
3) $\mathbb{R}^2 \setminus \{(0,0)\}$



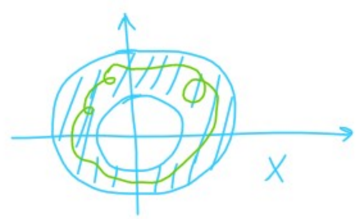
$\mathbb{R}^3 \setminus \{(0,0,0)\}$ јесте ПП

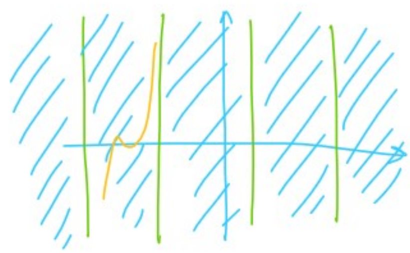
$\mathbb{R}^3 \setminus l$ није ПП

l – права



4)





② а) $2t\left(1+\sqrt{t^2-x}\right)dt - \sqrt{t^2-x}\,dx = 0$

б) $\left(1+x^2\sin 2t\right)dt - x\cos^2 t\,dx = 0$

в) $\left(tx^2+3t^2x\right)dt + \left(t^3+t^2x\right)dx = 0$

а) $M(t,x) = 2t\left(1+\sqrt{t^2-x}\right)$

$N(t,x) = -\sqrt{t^2-x}$

област: $t^2 - x \geq 0$

$x \leq t^2$

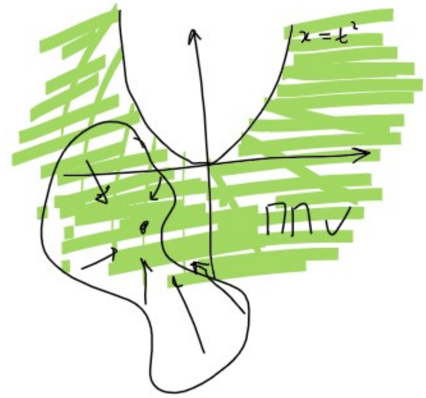
$$M'_x = 2t\cdot\frac{1}{2\sqrt{t^2-x}}(-1) = -\frac{t}{\sqrt{t^2-x}}$$

$$N'_t = -\frac{1}{2\sqrt{t^2-x}}\cdot 2t = -\frac{t}{\sqrt{t^2-x}} \quad \|$$

$$M'_x = N'_t + ПП$$

$$\Downarrow$$

тот. д.



$$\Rightarrow \exists F,\ \underbrace{dF}_{\frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx} = M dt + N dx \quad , F = ?$$

$$\frac{\partial F}{\partial t} = M = 2t(1+\sqrt{t^2-x})$$

$$\frac{\partial F}{\partial x} = N = -\sqrt{t^2-x} \quad \Big/ \int dx \quad \Rightarrow \quad F(t,x) = \frac{2}{3}(t^2-x)^{3/2} + c(t)$$

$$\frac{\partial F}{\partial t} = \sqrt{t^2-x}\cdot 2t + c'(t) = M = 2t(1+\sqrt{t^2-x})$$

$$\Rightarrow \underline{c'(t) = 2t} \Rightarrow \underline{c = t^2}$$

↳ мора само од t да зависи

$$F(t,x) = \frac{2}{3}(t^2-x)^{3/2} + t^2$$

ОР: $\frac{2}{3}(t^2-x)^{3/2} + t^2 = c, \quad c \in \mathbb{R}$

8 Интеграциони фактор

$M dt + N dx = 0 \qquad M'_x \neq N'_t$ није тот. д.

$$\underbrace{\mu\cdot(M dt + N dx) = 0}_{\text{тот. д.}}$$

често: $\mu = \mu(w(t,x))$, $\mu : \mathbb{R}^2 \to \mathbb{R}$, $w: \mathbb{R}^2 \to \mathbb{R}$, $\mu: \mathbb{R} \to \mathbb{R}$

предавања: $$\frac{\mu'(w)}{\mu(w)} = \frac{N'_t - M'_x}{w'_x\cdot M - w'_t\cdot N} \qquad \mu' = \frac{d\mu}{dw}$$

$$\int \frac{d\mu}{\mu} = \int \frac{N'_t - M'_x}{w'_x M - w'_t\cdot N}\, dw$$

→ зависи само од w

Циљ: наћи w!   Често: • $w(t,w) = at + bx$ $(w=t,\ w=x)$

• $w(t,x) = a\ln|t| + b\ln|x|$

• $w(t,x) = f(t) + g(x)$

• $w(t,x) = f(t)\cdot g(x)$

Проверити $\frac{1}{\mu} = 0$ за изгубљено решење

(3) а) $(q(t) - p(t)x)\,dt - dx = 0$, p, q познате неп. ← упоредити са ОР линеарне ДЈ $(w=t)$

б) $2tx\ln x\,dt + (t^2 + x^2\sqrt{x^2+1})\,dx = 0$

в) $x(2 - 3tx^2)\,dt - t(1 + tx^2)\,dx = 0$, на области $G = \{x>0, t>0\}$, решење пролази кроз $(2,1)$

г) $(\sqrt{t^2 - x} + 2t)\,dt - dx = 0$, μ у облику $\mu(t^2 - x)$ $(w = t^2 - x)$

д) $(t+2)\sin x\,dt + 2t\cos x\,dx = 0$ $(w = x)$

$\rightarrow \left((t+2)\sin x + 2t\cos x \cdot x' = 0 \ ; \ x' = -\frac{t+2}{2t}\cdot \operatorname{tg} x\right)$

б) $M(t,x) = 2tx\ln x$

$N(t,x) = t^2 + x^2\sqrt{x^2+1}$

$M'_x = 2t(1 + \ln x) \neq$

$N'_t = 2t$

од w!

$$\frac{d\mu}{\mu} = \frac{N'_t - M'_x}{w'_x\cdot M - w'_t\cdot N}\,dw = \frac{-2t\ln x}{w'_x\cdot(2tx\ln x) - \underbrace{\boxed{w'_t}\cdot(t^2 + x^2\sqrt{x^2+1})}_{0}}\,dw = \frac{-2t\ln x}{w'_x\cdot 2tx\ln x}\,dw$$

$w = w(x)$

$$= \frac{-dw}{x\cdot w'_x} =$$

$w'_t = 0$

$w = w(x)$

$$w = x \rightarrow = -\frac{dw}{x\cdot 1} = -\frac{dw}{w}$$

$$\int \frac{d\mu}{\mu} = \int -\frac{dw}{w}$$

$$\ln|\mu| = -\ln|w| \cancel{(+c)}$$

$$\vdots$$

$$\mu = \frac{1}{w} = \frac{1}{x}$$

$$\left[\ \frac{1}{\mu} = 0 \Rightarrow \frac{1}{x} = 0 \ \times\ \right]$$

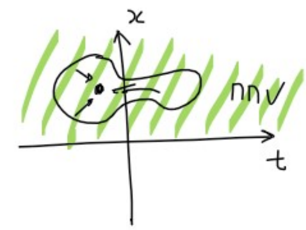
$$M\,dt + N\,dx = 0 \Big/ \cdot \frac{1}{x}$$

$$\underbrace{2t\ln x}_{\tilde{M}}\,dt + \underbrace{\left(\frac{t^2}{x} + x\sqrt{x^2+1}\right)}_{\tilde{N}}dx = 0$$

$\tilde{M}'_x = \tilde{N}'_t$ ✓

$x \neq 0$

$\underbrace{x > 0}$, $(\ln x)$



$$\Rightarrow \exists F, \quad dF = \tilde{M}\,dt + \tilde{N}\,dx$$

$$\left.\begin{array}{l} F'_t = \tilde{M} \\ F'_x = \tilde{N} \end{array}\right\} \ldots$$

ОР: $F(t,x) = t^2\ln x + \frac{1}{3}(x^2+1)^{3/2} = c,\ c\in\mathbb{R}$

в) $\underbrace{x(2-3tx^2)}_{M}dt \underbrace{-t(1+tx^2)}_{N}dx = 0$, на области $G=\{x>0, t>0\}$, решење пролази кроз $(2,1)$

$$M'_x = 2 - 9tx^2$$

$$N'_t = -1 - 2tx^2 \quad \#$$

$$\frac{d\mu}{\mu} = \frac{N'_t - M'_x}{w'_x\cdot M - w'_t\cdot N}dw = \frac{-3 + 7tx^2}{\underbrace{w'_x\cdot \boxed{x}}(2-3tx^2) + \underbrace{w'_t \boxed{t}}(1+tx^2)}dw = \frac{-3+7tx^2}{b(2-3tx^2)+a(1+tx^2)}dw$$

да се скрати

$$\Downarrow$$

$$w = a\ln|t| + b\ln|x| = a\ln t + b\ln x$$

$$w'_x = \frac{b}{x}$$

$$w'_t = \frac{a}{t}$$

$1:\quad -3 = 2b + a$

$tx^2:\quad 7 = -3b + a$

$b = -2,\ a = 1$

$$\Rightarrow w = \ln t - 2\ln x$$

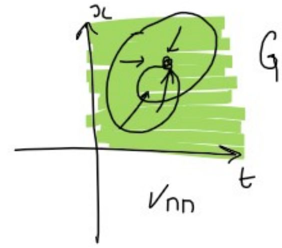
$$\Rightarrow \frac{d\mu}{\mu} = 1\,dw \Big/ \int$$

$$\Rightarrow \mu = e^w = e^{\ln t - 2\ln x} = \frac{t}{x^2}$$

$$\ldots = \frac{\ldots}{x^2}$$

$$/ \cdot \frac{t}{x^2} \Rightarrow \underbrace{\left(2\frac{t}{x} - 3tx^2\right)}_{\widetilde{M}} dt - \underbrace{\left(\frac{t^2}{x^2} + t^3\right)}_{\widetilde{N}} dx = 0$$

област:



$$\widetilde{M}'_x = \widetilde{N}'_t$$

$$\Downarrow$$

тот. д.

$$\left.\begin{aligned} \frac{\partial F}{dt} &= \widetilde{M} \\ \frac{\partial F}{\partial t} &= \widetilde{N} \end{aligned}\right\} \therefore$$

ОР:

$$F(t,x) = \frac{t^2}{x} - t^3 x = c, \; c \in \mathbb{R}$$

$$\frac{1}{\mu} = 0 \;, \quad \frac{x^2}{t} = 0 \Rightarrow x = 0$$

$0dt - \ldots d0 = 0 \; \checkmark$

$G \cap \{x = 0\} = \emptyset$

није реш.

Кроз $(2,1)$: $\quad t = 2$, $x = 1$

$$\frac{2^2}{1} - 2^3 \cdot 1 = c \Rightarrow c = -4$$

ПР: $\quad \frac{t^2}{x} - t^3 x = -4.$