2 - уторак 12.10.2021.

2 Хомогена ДЈ

$$x' = f\left(\frac{x}{t}\right) \quad , \quad f \in C(a,b)$$

смена: $x(t) \rightsquigarrow y(t)$

$$\left.\begin{aligned} y(t) = \frac{x(t)}{t} \quad &\Rightarrow \quad x = yt \ /' \\ & x' = (yt)' = y' \cdot t + y \cdot 1 = y + ty' \end{aligned}\right\} \rightsquigarrow \text{РП}$$

① а) $x' = e^{\frac{x}{t}} + \frac{x}{t}$

б) $x' = -\frac{x^2+t^2}{2xt}$

в) $x' = \frac{x^2 - 2xt - t^2}{x^2 + 2xt - t^2}$

г) $t \sin\frac{x}{t} \cdot x' = x \cdot \sin\frac{x}{t} + t$

В) $$x' = \frac{x^2 - 2xt - t^2}{x^2 + 2xt - t^2} \overset{|:t^2}{=} \frac{\left(\frac{x}{t}\right)^2 - 2\frac{x}{t} - 1}{\left(\frac{x}{t}\right)^2 + 2\frac{x}{t} - 1} = f\left(\frac{x}{t}\right)$$

смена: $y = \frac{x}{t} \Rightarrow x = yt \Rightarrow x' = y't + y$

$$y't + y = \frac{y^2 - 2y - 1}{y^2 + 2y - 1}$$

$$y't = \frac{y^2-2y-1}{y^2+2y-1} - y = \frac{y^2 - 2y - 1 - y^3 - 2y^2 + y}{y^2+2y-1} = \frac{-y^3 - y^2 - y - 1}{y^2+2y-1}$$

$\overline{0}$?

$$\frac{y^2+2y-1}{y^3+y^2+y+1} y' = -\frac{1}{t} \ \Big/ \int \quad (\text{РП})$$

$$-\ln|t| + c = \int \frac{y^2+2y-1}{y^3+y^2+y+1}\,dy \ , \ c \in \mathbb{R}$$

$$\frac{y^2+2y-1}{y^3+y^2+y+1} = \frac{A}{y+1} + \frac{By+C}{y^2+1} = \frac{A(y^2+1) + (By+C)(y+1)}{(y+1)(y^2+1)} = \frac{y^2(A+B) + y(B+C) + (A+C)}{(y+1)(y^2+1)}$$

$$y^3 + y^2 + y + 1 = (y+1)(y^2+1)$$

$$\left.\begin{aligned} A + B &= 1 \\ B + C &= 2 \\ A + C &= -1 \end{aligned}\right\}$$

$A = -1, B = 2, C = 0$

$A=-1, B=2, C=0$

$$-\ln|t|+c=\int\frac{-1}{y+1}\,dy+\int\frac{2y\,dy}{y^2+1}=-\ln|y+1|+\ln|y^2+1|=\ln\left|\frac{y^2+1}{y+1}\right|$$

e

$$\frac{e^c}{|t|}=\left|\frac{y^2+1}{y+1}\right| \Rightarrow \frac{c_1}{t}=\frac{y^2+1}{y+1}\,,\quad c_1\in\mathbb{R}\setminus\{0\}\ (c_1=\pm e^c)$$

$$\Big\downarrow\ y=\frac{x}{t}$$

$$\frac{c_1}{t}=\frac{x^2+t^2}{xt+t} \Rightarrow \boxed{c_1=\frac{x^2+t^2}{x+t}\,,\ c_1\in\mathbb{R}\setminus\{0\}\ (*)}$$

имплицитно задато решење

експлицитно: $x(t)=\ldots$

Да ли је $(y+1)(y^2+1)=0$? $\rightsquigarrow$ $y=-1$

$$\frac{x}{t}=-1 \Rightarrow \overset{(\#)}{x=-t}\ ?$$

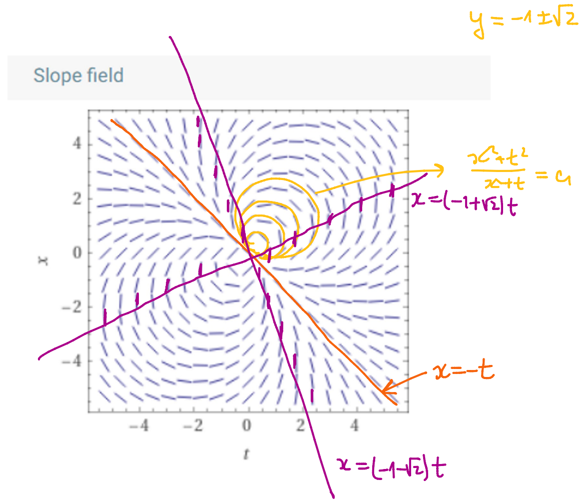
$$-1=\frac{t^2+2t^2-t^2}{t^2-2t^2-t^2}=\frac{2t^2}{-2t^2}=-1\ \checkmark$$

ОР: $(*)+(\#)$

НАП: почетак - делимо $\left.\begin{matrix}x^2+2xt-t^2\\ \shortparallel\\ 0?\end{matrix}\right\}$ $x'=\infty$ вертикални нагиб!

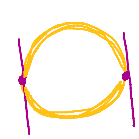
$y=\frac{x}{t}\,,\quad y^2+2y-1=0$

$y=-1\pm\sqrt{2} \rightsquigarrow x=(-1\pm\sqrt{2})t$



- решења као ф-је → искључујемо љубичасто
- решења као криве у $\mathbb{R}^2$

https://tinyurl.com/linkKaWA1



$\boxed{3}$ $x' = f\left(\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$, $f \in C(a, b)$

$\rightsquigarrow$ РП у ХОМ

1° $c_1 = c_2 = 0$: $x' = f\left(\frac{a_1 t + b_1 x}{a_2 t + b_2 x}\right) \overset{:t}{=} f\left(\frac{a_1 + b_1 \frac{x}{t}}{a_2 + b_2 \frac{x}{t}}\right) = g\left(\frac{x}{t}\right)$ $\to$ ХОМ

2° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, $a_1 b_2 - a_2 b_1 \neq 0$

$x(t) \rightsquigarrow v(u)$

$x = v + \alpha$, $\alpha, \beta \in \mathbb{R}$

$t = u + \beta$

$$\left.\begin{aligned} a_1 t + b_1 x + c_1 &= a_1(u+\beta) + b_1(v+\alpha) + c_1 = a_1 u + b_1 v + \underbrace{(a_1\beta + b_1\alpha + c_1)}_{=0}, \\ a_2 t + b_2 x + c_2 &= \ \text{---} \qquad = a_2 u + b_2 v + \underbrace{(a_2\beta + b_2\alpha + c_2)}_{=0}, \end{aligned}\right\}$$

$\exists_1$ решење система

$x' = \frac{dx}{dt}$?

$v' = \frac{dv}{du}$

$v = x - \alpha \Rightarrow \frac{dv}{dx} = 1$

$t = u + \beta \Rightarrow \frac{dt}{du} = 1$

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dv}{du} = \frac{dv}{dt} \cdot \frac{dt}{du} = \frac{dv}{dx} \cdot \underbrace{\frac{dx}{dt}}_{=x'} \cdot \frac{dt}{du} \Rightarrow v' = x'$$

$$v' = f\left(\frac{a_1 u + b_1 v}{a_2 u + b_2 v}\right) \rightsquigarrow 1^\circ$$

3° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \Rightarrow a_1 b_2 = b_1 a_2$

3.1° ако је неки (a_1, a_2, b_1, b_2) нула $\to$ лако $\rightsquigarrow \boxed{1}$, РП

3.2° $\frac{a_1}{a_2} = \frac{b_1}{b_2} =$

$= \frac{a_1 t + b_1 x}{a_2 t + b_2 x}$

$$\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2} = \frac{a_1}{a_2} + \frac{c}{a_2 t + b_2 x + c_2}$$

$$= \frac{a_1 t + b_1 x + \frac{a_1}{a_2} c_2 + \left(c_1 - \frac{a_1}{a_2} c_2\right)}{a_2 t + b_2 x + c_2} = \frac{a_1}{a_2} + \frac{c_1 - \frac{a_1}{a_2} c_2}{\boxed{a_2 t + b_2 x + c_2}} \rightsquigarrow \boxed{1}$$

$= \frac{a_1}{a_2}$

2. а) $(x+2t-2)x' = x-t-1$

б) $x' = \frac{t+x+4}{t+x-6}$

а) $x' = \frac{x-t-1}{x+2t-2}$ ← $x+2t-2=0$?

$$\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2+1 = 3 \neq 0$$

$t = u + \alpha$

$x = v + \beta$, $\alpha, \beta = ?$

$x - t - 1 = v + \beta - (u+\alpha) - 1 = \boxed{v-u} + \underbrace{(\beta - \alpha - 1)}_{=0}$

$x + 2t - 2 = v + \beta + 2(u+\alpha) - 2 = \boxed{v+2u} + \underbrace{(\beta + 2\alpha - 2)}_{=0}$

$\to \beta = \frac{1}{3}, \alpha = \frac{4}{3}$

$\frac{dv}{du} = \frac{dx}{dt}$

$$v' = \frac{v-u}{v+2u} = \frac{\frac{v}{u} - 1}{\frac{v}{u} + 2} \quad (\text{ХОМ})$$

$$\frac{v}{u} = w \Rightarrow v = wu \;/'_u \Rightarrow v' = w'u + w$$

$w(u)$

$$w'u + w = \frac{w-1}{w+2} \Rightarrow \dots \int \frac{w+2}{w^2+w+1} dw = \int -\frac{du}{u}$$

$\neq 0$?

$\vdots$

$w(u) \rightsquigarrow v(u) = u \cdot w(u)$

$\rightsquigarrow x(t)$

- $w^2 + w + 1 = 0$? X
  >0
- $x + 2t - 2 = 0$?
  $x = -2t + 2$ $\quad 0 \cdot (-2) = -3t + 3$ X

б) $x' = \frac{t+x+4}{\dots}$ $\quad \begin{vmatrix} 1 & 1 \end{vmatrix} = 0$

б) $x' = \dfrac{t+x+4}{t+x-6}$ $\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} = 0$

$\boxed{1}$ $\boxed{\times}$ $t + x(t) = y(t)$ ...

$\boxed{4}$ Линеарна ДЈ

$p, q : (a,b) \to \mathbb{R}$ непр.

$$x' + p(t)\,x(t) = q(t)$$

$$\to \text{ОР}: \boxed{x(t) = e^{-\int p\,dt}\cdot\left(C + \int e^{\int p\,dt}\cdot q\,dt\right),\ C\in\mathbb{R}}$$

$\int p\,dt$ је иста прим. ф-ја на оба места

$$\underbrace{C\cdot e^{-\int p\,dt}}_{\text{ОР хомогене}} + \underbrace{e^{-\int p\,dt}\cdot\int e^{\int p\,dt}\,q\,dt}_{\text{ПР нехомогене}}$$

$q \equiv 0$ → хомогена ЛДЈ 1. реда
$q \neq 0$ → нехомогена ЛДЈ 1. реда

(3) а) $tx' - x = t^3$

б) $x' + x = \dfrac{1}{1+e^{2t}}$

в) $x' - 2tx = 6te^{t^2}$

г) $tx' + ax + t^n = 0$, $a\in\mathbb{R}, n\in\mathbb{N}$

а) $tx' - x = t^3 \ /: t$

$x' - \dfrac{x}{t} = t^2$, $p(t) = -\dfrac{1}{t}$
$q(t) = t^2$

$|t| = \operatorname{sgn} t \cdot t$

$$e^{\int p\,dt} = e^{-\int \frac{dt}{t}} = e^{-\ln|t|} = \frac{1}{|t|} \quad (\cancel{+c})$$

$$\int e^{\int p\,dt}\cdot q\,dt = \int \frac{1}{|t|}\cdot t^2\,dt = \operatorname{sgn} t\cdot\int \frac{t^2}{t}\,dt = \operatorname{sgn} t\cdot\frac{t^2}{2} = \frac{t|t|}{2}$$

$$\text{ОР}: x(t) = |t|\cdot\left(c + \frac{t|t|}{2}\right) = c|t| + \frac{t\cdot|t|^2}{2} = c|t| + \frac{t^3}{2} = \underbrace{c_1 t + \frac{t^3}{2}},\ c_1\in\mathbb{R}$$

$c\in\mathbb{R}$