12 - уторак 21.12.2021.

① $x' = x(1-x)$. Без решавања:

а) Свако решење т.д. $x(0) = a \in (0,1) \Rightarrow (\forall t)\ 0 < x(t) < 1$.

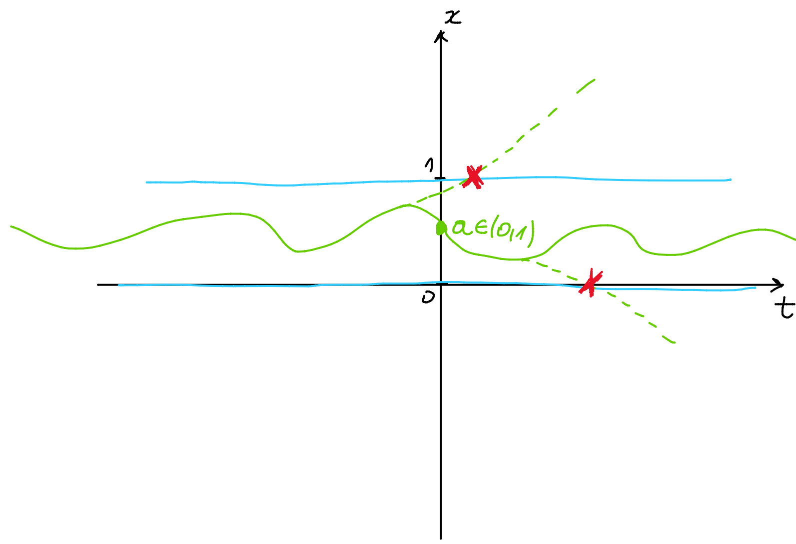
б) Наћи $\lim\limits_{t\to\infty} x(t)$ у зависности од $x(0) = a \in \mathbb{R}$.

а) $x' = x(1-x)$

$x \equiv 0$ ✓

$x \equiv 1$ ✓

$x(0) = a \in (0,1)$



$$\left.\begin{aligned} F(x,t) &= x(1-x) = x - x^2 \\ \frac{\partial F}{\partial x} &= 1 - 2x \in C(\mathbb{R}) \end{aligned}\right\} \Rightarrow F \in C^1(\mathbb{R})$$

$\Rightarrow$ Пикар

$\Rightarrow$ решења се не секу

$\Rightarrow$ $x(t)$ је у траци $\mathbb{R}\times(0,1) \Rightarrow (\forall t)\ 0 < x(t) < 1$.

б) $x(0) = a \in \mathbb{R}$

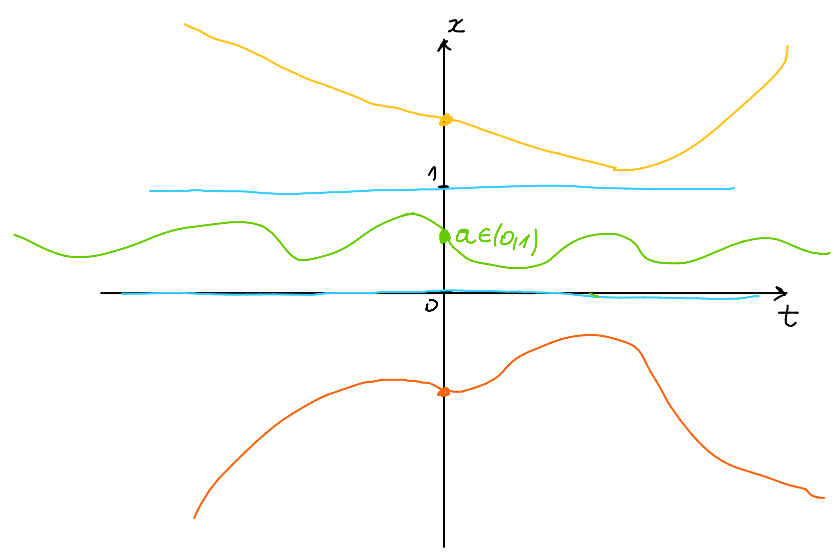
1° $a \in (0,1) \Rightarrow x(t) \in (0,1)$

2° $a = 0 \Rightarrow x(t) \equiv 0$

3° $a = 1 \Rightarrow x(t) \equiv 1$

4° $a > 1 \Rightarrow x(t) > 1$

5° $a < 0 \Rightarrow x(t) < 0$



1° $a \in (0,1) \Rightarrow x \in (0,1)$

$$x' = \underset{(0,1)}{\underset{\cap}{x}}\,\underbrace{(1-x)}_{\in(0,1)} \in (0,1) \Rightarrow x' > 0 \Rightarrow x \nearrow$$

$x \nearrow$, $x \in C^1$, x огр. са 1 $\Rightarrow$ x има хор. асимпт.

$\Rightarrow \exists \lim x \Rightarrow \exists \lim x' \Rightarrow x'(t) \xrightarrow[t\to\infty]{} 0$

$0 = \lim x(1-x) \Rightarrow x \to 0 \lor x \to 1$ због $x \nearrow$

$\Rightarrow \lim\limits_{t\to\infty} x(t) = 1.$

4° домаћи $\to \lim\limits_{t\to\infty} x(t) = 1$

5° $a<0,\ x<0$

$x' = \underbrace{x}_{<0}\,\underbrace{(1-x)}_{>0} < 0 \Rightarrow x \searrow$

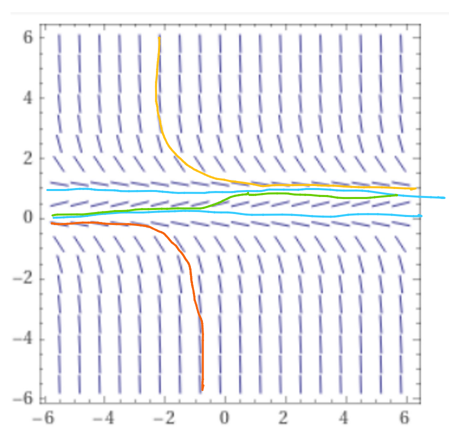
[ако x има хор. асимпт. $\Rightarrow \exists \lim x \Rightarrow \exists \lim x' \Rightarrow x' \to 0$

$\Rightarrow x \to 0 \lor x \to 1$

не може $x \searrow$; не може] $\lightning$

$\Rightarrow x \xrightarrow{t\to\infty} -\infty$

$x' \to -\infty$



$$\lim_{t\to\infty} x(t) = \begin{cases} 1, & a>0 \\ 0, & a=0 \\ -\infty, & a<0 \end{cases}$$

② Доказати да $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = \left(\sqrt{x^2+y^2}, \sqrt[4]{x^2+y^2}\right)$ није локално Липшицова ни у једној околини тачке $(0,0)$.

Рј. $X' = f(X)$, $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$\Updownarrow$

$x' = \sqrt{x^2+y^2}$

$y' = \sqrt[4]{x^2+y^2}$

Не важи Пикар за $X(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\|f(X_1)-f(X_2)\| \le L\cdot\|X_1-X_2\| \quad , \left.\begin{matrix} X_1=(x_1,y_1) \\ X_2=(x_2,y_2)\end{matrix}\right\} \text{у околини } (0,0)$$

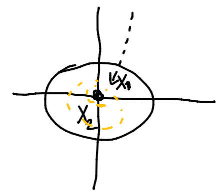
Узмемо $X_2=(0,0) \rightsquigarrow f(X_2)=(0,0)$

$\|\cdot\|$ - узимамо еуклидску норму

$$\|f(X_1)\|\le L\cdot\|X_1\|$$

$$\|(\sqrt{x_1^2+y_1^2}, \sqrt[4]{x_1^2+y_1^2})\| \le L\cdot\|(x_1,y_1)\|$$

$$\sqrt{(\sqrt{x_1^2+y_1^2})^2+(\sqrt[4]{x_1^2+y_1^2})^2} \le L\cdot\sqrt{x_1^2+y_1^2} \ \Big/^2$$



$$x_1^2+y_1^2+\sqrt{x_1^2+y_1^2} \le L^2\cdot(x_1^2+y_1^2) \ \Big/ :(x_1^2+y_1^2)$$

$X_1\ne(0,0)$

али $X_1\to(0,0)$

$$\left.1+\underbrace{\boxed{\frac{1}{\sqrt{x_1^2+y_1^2}}}}_{\to\infty}\le L^2 \ \right\} \ \infty \le L^2 \in \mathbb{R}^2 \ ↯$$

$$\lim_{(x_1,y_1)\to(0,0)} \frac{1}{\sqrt{x_1^2+y_1^2}}=\infty$$

(3) Формирати низ итерација из доказа Пикарове Т за проблем:

а) $x'=\frac{x}{t}$, $x(t_0)=x_0$, $t_0>0$

б) $X'=AX$, $A\in M_n(\mathbb{R})$, $X(0)=X_0$

$$\mathbf{x}_0(t)\equiv\mathbf{x}_0,\quad \mathbf{x}_{n+1}(t):=\mathbf{x}_0+\int_{t_0}^t F(\mathbf{x}_n(s),s)ds,$$

б) $X_0(t)=X_0$ $\qquad F(X,t)=A\cdot X$

$$X_1(t)=X_0+\int_0^t AX_0(s)\,ds = X_0+\int_0^t AX_0\,ds = X_0+AX_0\cdot s\Big|_0^t = X_0+tAX_0$$

$$X_2(t)=X_0+\int_0^t A\cdot X_1(s)\,ds = X_0+\int_0^t A(X_0+sAX_0)\,ds = X_0+tAX_0+\frac{s^2}{2}A^2X_0\Big|_0^t = X_0+tAX_0+\frac{t^2}{2}A^2X_0$$

$$X_3(t)=X_0+\int_0^t A\cdot X_2(s)\,ds = X_0+\int_0^t A\cdot\left(X_0+sAX_0+\frac{s^2}{2}A^2X_0\right)ds = X_0+tAX_0+\frac{t^2}{2}A^2X_0+\frac{t^3}{6}A^3X_0$$

$$\vdots$$

индукцијом: $X_n(t)=\sum^{n}\frac{t^k}{k!}\cdot A^k\cdot X_0$

индукцијом: $X_n(t)=\sum_{k=0}^{n}\frac{t^k}{k!}\cdot A^k\cdot X_0$
(доказати)

$$X_\infty(t)=\lim_{n\to\infty}X_n(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}A^kX_0=e^{tA}\cdot X_0$$

д) $x_0(t)\equiv x_0$, $F(x,t)=\frac{x}{t}$

$$x_1(t)=x_0+\int_{t_0}^{t}F(x_0(s),s)\,ds=x_0+\int_{t_0}^{t}\frac{x_0(s)}{s}\,ds=x_0+\int_{t_0}^{t}\frac{x_0}{s}\,ds=x_0+x_0\ln\frac{t}{t_0}.$$

доказати: $x_0, x_1, x_2, x_3, \dots$

$x_n(t)=\sum_{k=0}^{n}\frac{x_0}{k!}\ln^k\frac{t}{t_0}$

помоћ: $\int_{t_0}^{t}\frac{\ln^k\frac{s}{t_0}}{s}=\dots=\frac{\ln^{k+1}\frac{t}{t_0}}{k+1}$.

$x_\infty(t)=\frac{x_0}{t_0}\cdot t$

Токови

$$\frac{d}{dt}\phi^{t}(\underbrace{x}_{\in\mathbb{R}^n})=F(\phi^t(x),t)\quad,\ \phi^0=id$$

време

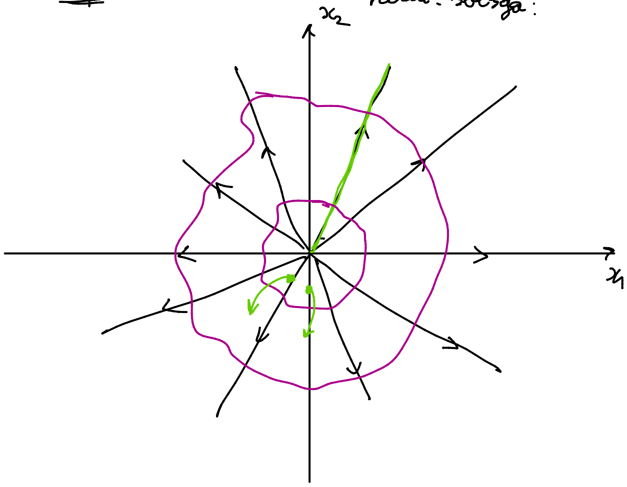
посматрамо трансформацију простора ($\mathbb{R}^n$)

ϕ^t називамо током век. поља F

$\phi^t:\mathbb{R}^n\to\mathbb{R}^n$
($U\subseteq\mathbb{R}^n$, $\phi^t: U\to\mathbb{R}^n$)

пр.

нестаб. звезда:



$$\begin{bmatrix}x_1'\\x_2'\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

ОР: $x_1=c_1e^t$
$x_2=c_2e^t$

$\phi^t\underbrace{(x_1,x_2)}_{\in\mathbb{R}^2}=(x_1e^t,x_2e^t)$, $\phi^t:\mathbb{R}^2\to\mathbb{R}^2$

пр. $X' = AX \rightsquigarrow \underbrace{\phi^t(X_0) = e^{tA} \cdot X_0 \text{ је ток}}$, $F(X,t) = A\cdot X$

доказ: 1) $\frac{d}{dt}\phi^t(X_0) = F(\phi^t(X_0), t)$

$$\left.\begin{aligned} &\frac{d}{dt}(e^{tA}X_0) = Ae^{tA}\cdot X_0 \\ &F(\phi^t(X_0), t) = A\cdot\phi^t(X_0) = A\cdot e^{tA}\cdot X_0 \end{aligned}\right\} = \checkmark$$

2) $\phi^0 = id$

$\phi^0(X_0) = e^{0A}\cdot X_0 = E\cdot X_0 = X_0$ ✓

4) Наћи токове за ДЈ:

а) $x' = x + 2t$

б) $x' = -x$
$y' = x^2 + y$

а) ОР: $x(t) = ce^t - 2t - 2$

$x(0) = c - 2\cdot 0 - 2 = c - 2 = x_0$

$c = 2 + x_0$

$\phi^t(x_0) = (2+x_0)e^t - 2t - 2$

провера: 1) $\frac{d}{dt}\phi^t(x_0) = \frac{d}{dt}\left((2+x_0)e^t - 2t - 2\right) = (2+x_0)e^t - 2$

$F(\phi^t(x_0), t) = \phi^t(x_0) + 2t = (2+x_0)e^t - 2t - 2 + 2t$ $\|$ ✓

2) $\phi^0(x_0) = 2 + x_0 - 2 = x_0$ ✓

б) $x' = -x \Rightarrow x(t) = c_1 e^{-t}$

$y' = x^2 + y = c_1^2 e^{-2t} + y \Rightarrow y(t) = c_2 e^t - \frac{1}{3}c_1^2 e^{-2t}$

$x(0) = x_0 \Rightarrow c_1 = x_0$

$y(0) = y_0 \Rightarrow y_0 = c_2 - \frac{1}{3}x_0^2 \Rightarrow c_2 = y_0 + \frac{1}{3}x_0^2$

$\phi^t(x_0, y_0) = \left(x_0 e^{-t}, (y_0 + \frac{1}{3}x_0^2)e^t - \frac{1}{3}x_0^2 e^{-2t}\right)$

Једнопараметарска фамилија пресликавања (ЈФП)

Једнопараметарска фамилија пресликавања (ЈФП)

$$\varphi^t : M \to M,\ \forall t \in \mathbb{R} \qquad (\mathbb{R}^n = M)$$

1) $\varphi^{t+s} = \varphi^t \circ \varphi^s$

2) $\varphi^0 = \mathrm{id}$

$\Longleftrightarrow$ дејство групе $(\mathbb{R},+)$ на M

⑤ Проверити да су дате ЈФП у $\mathbb{R}^n$:

а) $\varphi^t(x) = (t+1)\cdot x$

б) $\psi^t(x) = e^t \cdot x$

в) $\theta^t(x) = t\cdot \underbrace{(1,\dots,1)}_{n} + x$

в) 1) $\theta^{t+s} = \theta^t \circ \theta^s$

$$\theta^{t+s}(x) = (t+s)\cdot(1,\dots,1) + x$$

$$\theta^t \circ \theta^s(x) = \theta^t\big(s\cdot(1,\dots,1)+x\big) = t\cdot(1,\dots,1) + s\cdot(1,\dots,1) + x = (t+s)\cdot(1,\dots,1) + x \ \Big\} = \ \checkmark$$

2) $\theta^0(x) = 0\cdot(1,\dots,1) + x = x \Rightarrow \theta^0 = \mathrm{id}$ ✓

⑥ Испитати да ли су токови из зад. ④ ЈФП:

а) $\varphi^t(x_0) = (2+x_0)e^t - 2t - 2$

б) $\varphi^t(x_0,y_0) = \left(x_0 e^{-t},\ (y_0 + \frac{1}{3}x_0^2)e^t - \frac{1}{3}x_0^2 e^{-2t}\right)$ → домаћи (ДЗ)

а) $\varphi^t \circ \varphi^s(x_0) = (2+\varphi^s(x_0))e^t - 2t - 2 = \big(2 + (2+x_0)e^s - 2s - 2\big)e^t - 2t - 2 =$

$$= \underbrace{2e^t}_{\times} + (2+x_0)e^s\cdot e^t - 2se^t \underbrace{-2e^t}_{\times} - 2t - 2 =$$

$$= (2+x_0)e^{s+t} - 2se^t - 2t - 2$$

$$\not\parallel$$

$$\varphi^{t+s}(x_0) = (2+x_0)e^{t+s} - 2(t+s) - 2 = (2+x_0)e^{t+s} - 2t - 2s - 2$$

$\Rightarrow$ није ЈПФ

$\boxed{\text{T}}$ φ^t је ЈФП $\Longleftrightarrow$ F аутономно а) није аутономно

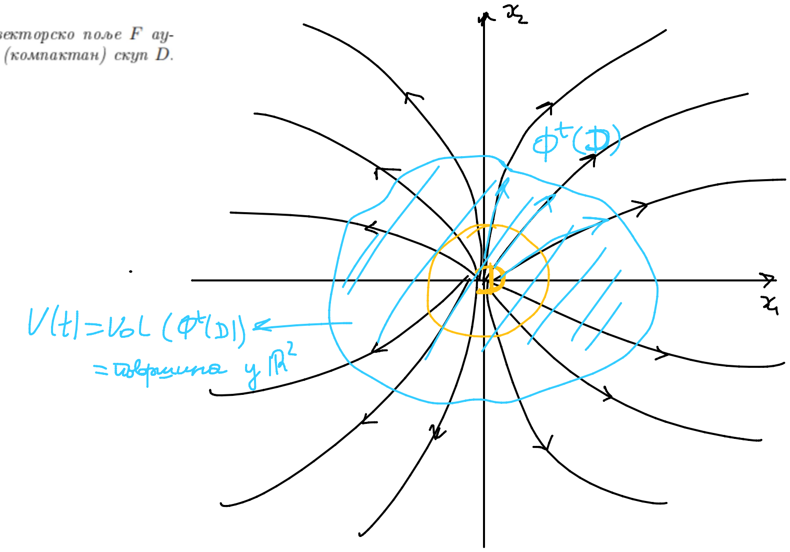
б) јесте

Лиувилова Т

Теорема 104. **(Лиувилова теорема - јача верзија.)** *Нека је векторско поље F аутономно, ϕ^t решење система (68) и $V(t) := \mathrm{Vol}(\phi^t(D))$, за неки мерљив (компактан) скуп D. Тада је*

$$\frac{dV(t)}{dt} = \int \cdots \int_{\phi^t(D)} \operatorname{div} F dy_1 \cdots dy_n,$$

где је $\operatorname{div} F = \nabla \cdot F$ *дивергенција векторског поља* F.



$\operatorname{div} F > 0 \;(=, <) \Rightarrow \frac{dV(t)}{dt} > 0 \;(=, <) \Rightarrow$ запремина се повећава (чува, смањује)

спец: $X' = AX$, $F(X) = A \cdot X$

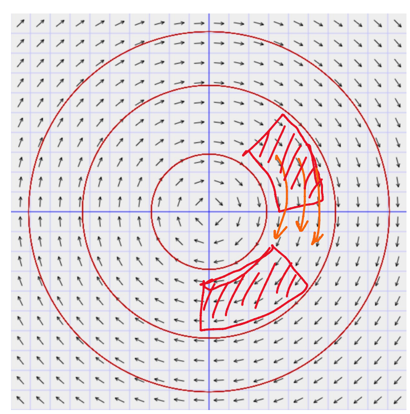
$$\operatorname{div}(F) = \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n} = \operatorname{tr} A$$

промена запр. зависи само од $\operatorname{tr} A$

пр. 1) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ – нестаб. чвор

$\operatorname{tr} A = 3 > 0 \Rightarrow Vol \nearrow$

2) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ – центар



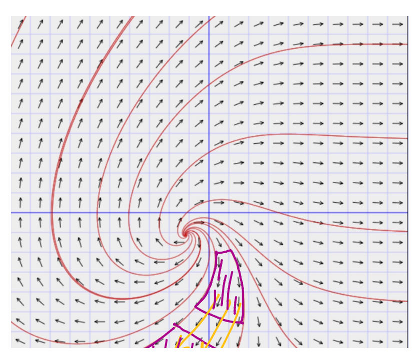
$\operatorname{tr} A = 0 \Rightarrow Vol$ се чува

(чак је изометрија ϕ^t)

$\phi^t = R_t$

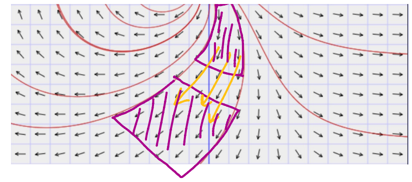
3) $x_1' = e^{x_1} + x_2$

$x_2' = x_2 - x_1$



$F(x_1, x_2) = (e^{x_1} + x_2, x_2 - x_1)$

$\operatorname{div} F = e^{x_1} + 1 > 0 \rightarrow$ повећава запр.



4) $x' = x^2+1$ $\rightsquigarrow$ $\frac{x'}{x^2+1} = 1 \,/\int$ $\Rightarrow$ $x = tg(t+c)$

(у $\mathbb{R}^1$)

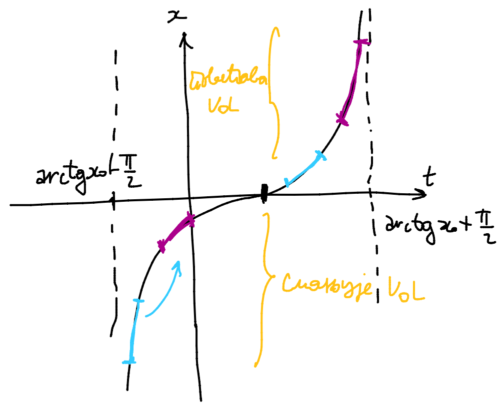
$$\phi^t(x) = tg(t + arctg\, x_0)$$

$F(x) = x^2+1$

$div F = (x^2+1)'_x = 2x$

$x>0 \to Vol \nearrow$

$x<0 \to Vol \searrow$



Vol у $\mathbb{R}^1$ $\Leftrightarrow$ дужина (лука)

Векторска поља и комутатор (у $\mathbb{R}^n$)

$$F = (F_1, \dots, F_n) \leftrightsquigarrow F = F_1\frac{\partial}{\partial x_1} + F_2\frac{\partial}{\partial x_2} + \dots + F_n\frac{\partial}{\partial x_n}$$

$$\left(\frac{\partial}{\partial x_k} = (0,0,\dots,\overset{\overset{k}{\downarrow}}{1},\dots,0) \right)$$

$F: C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$

$$F(f) = F_1\frac{\partial f}{\partial x_1} + \dots + F_n\frac{\partial f}{\partial x_n}$$

- $F(\lambda f + \mu g) = \lambda F(f) + \mu F(g)$
- $F(fg) = F(f)g + fF(g)$

Комутатор: $\underbrace{[F,G]}_{\text{век. поље}} = F\circ G - G\circ F$

$$[F,G](\underset{\underset{C^\infty}{\Uparrow}}{f}) = \underbrace{F(\underbrace{G(f)}_{C^\infty})}_{C^\infty} - \underbrace{G(\underbrace{F(f)}_{C^\infty})}_{C^\infty}$$

$[F,G] = 0 \Leftrightarrow F\circ G = G\circ F$ (комутирају)

смена координата: $\varphi: (x_1,\dots,x_n) \longrightarrow (y_1,\dots,y_n)$

$$(\forall k)\quad \frac{\partial}{\partial y_k} = \sum_{l=1}^{n} \frac{\partial x_l}{\partial y_k} \cdot \frac{\partial}{\partial x_l}$$

$$\varphi_* F = G \qquad F \text{ у } x\text{-коорд.} \qquad G \text{ у } y\text{-коорд.}$$

push-forward
(гурање)

(7) $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ у $\mathbb{R}^2$

Нађи ток ϕ^t овог в.п. X

$$\frac{d}{dt}\phi^t = X(\phi^t) \quad , \quad \phi^0 = \mathrm{id}$$

$$\phi^t(x) = (u(t), v(t)) \qquad X \text{ израчунато у } \phi^t = (u,v)$$

$$(u'(t), v'(t)) = -v(t)\frac{\partial}{\partial x} + u(t)\frac{\partial}{\partial y} = (-v(t), u(t))$$

$$\left.\begin{aligned} u' &= -v \\ v' &= u \end{aligned}\right\} \quad \begin{aligned} u &= c_1 \sin t + c_2 \cos t \\ v &= -c_1 \cos t + c_2 \sin t \end{aligned}$$

$$\phi^0(x_0) = x_0 = (x_1, x_2)$$

[иначе $\phi^t = R_t \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (R_{-t})]

$$u(0) = c_2 = x_1$$
$$v(0) = -c_1 = x_2$$

$$\Rightarrow \quad \phi^t(x_1, x_2) = (-x_2 \sin t + x_1 \cos t,\ x_2 \cos t + x_1 \sin t)$$

(8) Нађи в.п. за ток $\phi^t(x,y) = (\underbrace{x\cos t + y \sin t}_{f_1}, \underbrace{-x \sin t + y\cos t}_{f_2})$

$$\frac{d}{dt}\phi^t(x,y) = X(\phi^t(x,y)) \qquad , \quad \phi^0 = \mathrm{id}\ ✓$$

?

$$\frac{d}{dt}\phi^t(x,y) = (\underbrace{-x\sin t + y\cos t}_{f_2}, \underbrace{-x\cos t - y \sin t}_{-f_1}) = X(\overset{(f_1,f_2)}{\phi^t(x,y)})$$

$$X(f_1, f_2) = (f_2, -f_1)$$

$$X(x,y) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

(9) $X = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 2x_3\frac{\partial}{\partial x_3}$, $\phi^t(x_1,x_2,x_3) = (-x_2 \sin t + x_1\cos t,\ x_2\cos t + x_1 \sin t,\ e^t x_3)$

а) $X \rightsquigarrow \psi^t$

б) Да ли ψ и ϕ комутирају? $(\psi^t \circ \phi^s = \phi^s \circ \psi^t)$

$$\begin{tikzcd} \bullet \arrow[r, "\phi^s"] \arrow[d, "\psi^t"'] \arrow[dr, dashed] & \bullet \arrow[d, "\psi^t"] \\ \bullet \arrow[r, "\phi^s"'] & \bullet \end{tikzcd}$$

а) $\psi^t = (\underline{u}, \underline{v}, \underline{w})$

$$u' = u$$
$$v' = v \qquad \therefore \qquad \psi^t(x_1, x_2, x_3) = (x_1 e^t, x_2 e^t, x_3 e^{2t})$$
$$w' = 2w$$

б) $\psi^t \circ \phi^s(x_1, x_2, x_3) = \psi^t(-x_2 \sin s + x_1 \cos s, x_2 \cos s + x_1 \sin s, e^s x_3) =$

$= (-x_2 \sin s\, e^t + x_1 \cos s\, e^t, x_2 \cos s\, e^t + x_1 \sin s\, e^t, e^s x_3 e^{2t})$ ✓

$\phi^s \circ \psi^t(x_1, x_2, x_3) = \phi^s(x_1 e^t, x_2 e^t, x_3 e^{2t}) =$

$= (-x_2 e^t \sin s + x_1 e^t \cos s, x_2 e^t \cos s + x_1 e^t \sin s, e^s x_3 e^{2t})$ ✓

$\Rightarrow$ ψ и ϕ комутирају

домаћи: Наћи в.п. Y за ϕ^t и израчунати $[X, Y]$

$\boxed{T}$ $F \rightsquigarrow \phi^t$ $\qquad [F, G] = 0 \iff \phi$ и ψ комутирају

$G \rightsquigarrow \psi^t$

(10) $X_1 = \frac{\partial}{\partial x} \qquad Y_1 = \frac{\partial}{\partial y}$

$X_2 = \frac{\partial}{\partial x} \qquad Y_2 = (1 + x^2) \frac{\partial}{\partial y}$

Наћи $[X_1, Y_1]$, $[X_2, Y_2]$.

$$[X_1, Y_1](f) = X_1(Y_1(f)) - Y_1(X_1(f)) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = 0$$

2. Шварц. изв. функције.

$[X_2, Y_2] = \frac{\partial}{\partial x}\Big(\overbrace{\quad}^{1+x^2}\overbrace{\quad}^{\frac{\partial f}{\partial y}}\Big) \ldots$

2. Дефу. изв. функције.

$$[X_2, Y_2] = \frac{\partial}{\partial x}\Big(\overbrace{(1+x^2)}^{1+x^2}\overbrace{\tfrac{\partial}{\partial y} f}^{\frac{\partial f}{\partial y}}\Big) - (1+x^2)\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) =$$

$$= \underbrace{\frac{\partial}{\partial x}(1+x^2)}_{=\,2x}\frac{\partial f}{\partial y} + \underbrace{(1+x^2)\frac{\partial^2 f}{\partial x \partial y}}_{\times} - \underbrace{(1+x^2)\frac{\partial^2 f}{\partial y \partial x}}_{\times} = 2x\frac{\partial f}{\partial y}$$

Лајбн. пр.

$$\Rightarrow [X_2, Y_2] = 2x\frac{\partial}{\partial y} = (0, 2x)$$

(11) Доказати $[X, fY] = X(f)\cdot Y + f\cdot[X,Y]$, $f \in C^\infty(\mathbb{R}^n)$

$$[X, fY](g) = X(\underbrace{f}_{C^\infty}\underbrace{Y(g)}_{C^\infty}) - fY(X(g)) = X(f)Y(g) + f\,X(Y(g)) - f\,Y(X(g)) =$$

$$= X(f)\cdot Y(g) + f\cdot\big(X(Y(g)) - Y(X(g))\big) = X(f)\cdot Y(g) + f\cdot[X,Y](g)$$

$$\Rightarrow [X, fY] = X(f)Y + f\cdot[X,Y]$$

(12) $\{xy>0\} \subseteq \mathbb{R}^2$, $X = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

$Y = 2\sqrt{xy}\,\frac{\partial}{\partial x}$

а) $[X,Y]$

б) $x = uv^2$

$y = u$

изразити X и Y у новим коорд.

а) дакле – доста посла

$$[X,Y] = X(Y) - Y(X)$$

$$X(Y) = \left(\frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(2\sqrt{xy}\,\frac{\partial}{\partial x}\right) =$$

$$= \underbrace{\frac{x}{y}\frac{\partial}{\partial x}\Big(\underbrace{2\sqrt{xy}}\,\underbrace{\tfrac{\partial f}{\partial x}}\Big)}_{} + \frac{\partial}{\partial y}\Big(\underbrace{2\sqrt{xy}}\,\underbrace{\tfrac{\partial}{\partial x}}\Big) =$$

$$= \underbrace{\frac{x}{y}\frac{\partial}{\partial x}(2\sqrt{xy})\frac{\partial}{\partial x} + \frac{x}{y}2\sqrt{xy}\frac{\partial^2}{\partial x^2}}_{} + \ldots \qquad = \ldots = 0$$

б) $\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\cdot\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\cdot\frac{\partial}{\partial y} =$

$= v^2\frac{\partial}{\quad} + \frac{\partial}{\quad}$

$x = uv^2$
$y = u$ $\Big\}$ $\frac{x}{y} = v^2 \Rightarrow v = \sqrt{\frac{x}{y}}$

$u = y$

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} =$$

$$= v^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} =$$

$$= \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = X$$

$$\left.\begin{aligned} x &= uv^2 \\ y &= u \end{aligned}\right\} \quad \frac{x}{y} = v^2 \Rightarrow v = \sqrt{\frac{x}{y}}, \quad u = y$$

$$\frac{\partial x}{\partial u} = v^2$$

$$\frac{\partial y}{\partial u} = 1$$

$$\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} =$$

$$= 2uv\frac{\partial}{\partial x} = 2\cdot y \cdot \sqrt{\frac{x}{y}}\frac{\partial}{\partial x} = 2\sqrt{xy}\frac{\partial}{\partial x} = Y$$

$$\Rightarrow \quad \varphi(x,y) = \Big(\underset{u}{y}, \underset{v}{\sqrt{\tfrac{x}{y}}}\Big) \qquad \varphi_* X = \frac{\partial}{\partial u}$$

$$\varphi_* Y = \frac{\partial}{\partial v}$$

$$\varphi_*[X,Y] \underset{\text{пред.}}{=} [\varphi_* X, \varphi_* Y] = \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right] = 0 \quad \Rightarrow [X,Y] = 0$$

(13) Да ли ∃ смена од X_2, Y_2 из зад (10) буду координатна?

$$X_2 = \frac{\partial}{\partial x} \qquad Y_2 = (1+x^2)\frac{\partial}{\partial y}$$

$$\left.\begin{aligned} \varphi_* X_2 &= \frac{\partial}{\partial u} \\ \varphi_* Y_2 &= \frac{\partial}{\partial v} \end{aligned}\right\} \; ? \quad \varphi ?$$

$$[X_2, Y_2] = 2x\frac{\partial}{\partial y}$$

$\nexists$

пп да ∃: $\varphi_*[X_2, Y_2] = [\varphi_* X_2, \varphi_* Y_2] = \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right] = 0 \Rightarrow [X_2, Y_2] = 0$

(14) $X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$

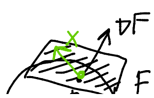
$Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$ у $\mathbb{R}^3$

$Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

сфера $S^2 = \{x^2+y^2+z^2 = 1\} \subseteq \mathbb{R}^3$

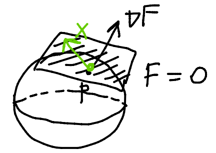
а) X, Y, Z тангентни на сферу

б) $[X,Y], [Y,Z], [X,Z]$ танг. на сферу → домаћи

в) 

$p \in S^2$, T_pS^2 – тангентни простор у p на S^2

2)



$p\in S^2$, T_pS^2 – тангентни простор у p на S^2

$\nabla F \perp T_pS^2$

$$F(x,y,z)=x^2+y^2+z^2-1$$

$$X\perp \nabla F \rightsquigarrow \langle X,\nabla F\rangle = 0$$
(Y,Z)

$$\nabla F=(2x,2y,2z)$$
$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = (0,-z,y)$$
$$\langle X,\nabla F\rangle = \langle (0,-z,y),(2x,2y,2z)\rangle = 0-2yz+2yz=0 \ \checkmark$$

$\vdots$

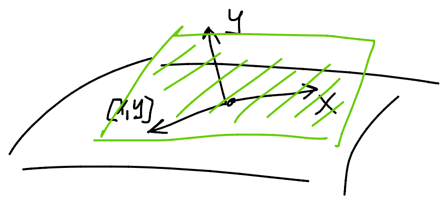
15) $X=\dfrac{\partial}{\partial x}$

$Y=\dfrac{\partial}{\partial y}+x\dfrac{\partial}{\partial z}$ у $\mathbb{R}^3$. Доказати да $\nexists$ површ $S\subseteq\mathbb{R}^3$ т.д. $X,Y\in TS$.
↳ тангентни на површ

TS – ознака за тангентни простор површи S, $TS=\bigsqcup_{p\in S} T_pS$ – унија свих тангентних равни.

$\boxed{T}$ $X,Y\in TS \Rightarrow [X,Y]\in TS$, S-површ (димензије 2) [последица Фробенијусове Т – видети предавања]

$$[X,Y]=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}\right)-\left(\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}\right)=1\frac{\partial}{\partial z}+x\frac{\partial^2}{\partial x\partial z}-x\frac{\partial^2}{\partial z\partial x}=\frac{\partial}{\partial z}$$



S-површ (описује се са 2 параметра)
$\dim S=2$
$\dim T_pS=2$

$X,Y,[X,Y]\in T_pS$

$X=(1,0,0)$
$Y=(0,1,x)$
$[X,Y]=(0,0,1)$

$$\operatorname{rang}(X,Y,[X,Y])=\begin{bmatrix}1&0&0\\0&1&x\\0&0&1\end{bmatrix}=3$$

$\Rightarrow$ лин. нез. $\Rightarrow$ не могу да се сместе у раван T_pS која је в.п. дим 2.
(афини)